Four Point Gauss Quadrature Runge – Kuta Method Of Order 8 For Ordinary Differential Equations

AGAM, S.A
Department of Mathematics,Nigerian Defence Academy (NDA) Kaduna, Nigeria.
E-mail: agamsylvester@gmail.com.

Abstract : *We present a strong convergence implicit Runge-Kutta method, with four stages, for solution of initial value problem of ordinary differential equations. Collocation method is used to derive a continuous scheme; and the continuous scheme evaluated at special points, the Gaussian points of fourth degree Legendre polynomial, gives us four function evaluations and the Runge-Kutta method for the iteration of the solutions. Convergent properties of the method are discussed. Experimental problems used to check the quality of the scheme show that the method is highly efficient, A – stable, has simple structure, converges to exact solution faster and better than some existing popular methods cited in this paper.*

Keywords: *strong convergent, collocation method, Gaussian points, highly efficient, A – stable, simple structure,*

I. Introduction

A number of practical and real-life problems are models of ordinary differential equations (ODEs). Most of these are non-linear, stiff or oscillatory problems. Examples of such are found in Electric circuits systems, chemical reaction problems, population growth, Biology, Social Sciences etc. Explicit Runge-Kutta methods cannot efficiently handle them as they lack sufficient stability region. The instability of explicit Runge-Kutta (RK) methods motives the development of implicit methods (see 5). The coefficient of explicit method is a lower triangular matrix and that of the implicit is a square matrix. We have different types of implicit methods; Radau, Lobatto, Gauss Legendre quadrature. We have the Gauss-quadrature method of order four and six respectively (see 2). In this paper, in other to get higher and more stable method, we used four Gaussian points of Legendre polynomial to derive a four stage Gauss quadrature R-K method of order 8.which more efficient and easy in their implementation than the existing ones.

II. Basic Definitions/Preliminaries:

(i) A Runge-Kutta implicit scheme for first order ODEs is defineds as

$$\left.\begin{aligned} Y_{n+1} &= y_n + h\sum_{i=1}^{5} b_i F_i\,, \quad i = 1,2,\ldots S \ (S = \text{stages}) \\ \text{where } &= f(x_n + c_{ih}, \quad y_i), \quad y_n + \sum_{i=1}^{5} a_{ij} f_i \end{aligned}\right\} \qquad 1.01$$

(ii) Order of scheme: the order of implicit Runge-Kutta methods whose abscissae or integration Rodes are Gaussian points of Legendre polynomial of order p = (2S)

(iii) Error /error construct:

A Runge-Kutta solution can be expanded into Taylors series, the solution agrees with the Taylors series expander up to order P, the truncation error is the O(h^{p+1}) term. The error and error constant is associated with a linear difference operator $L(y(x),h)$. as $L(y(x),h) = y(x_{n+1}) - y_{n+1}$

$= \quad Co\ y(x_n) + C_1 h_{y^I}(x_n) + (2h^2 y^{II}(x_n) + \ldots C_p h^p y^{(p)}(x_n) + C_{p+1} h^{(n+1)} y(x_n)$

where $y(x_{n+1})$ and y_{n+1} are exact and Runge – Kutta approximate solutions, respectively.

According to (5) the method is of order P, with error constant C_{p+1} of $= C_1 = C_0 = C_2 \ldots C_p = 0$ and $C_{p+1} \neq 0.$

(iv) The three stage implicit Gauss-quadrature (3) of order 6 is summarized by the table below:

1A. Buchers Coefficient table (1)
A $=a_{ij}$

0	0	0	0	0	0
$\frac{1}{2} - \frac{\sqrt{3}}{4}$	$-\left(\frac{11}{320} - \frac{3\sqrt{3}}{80}\right)$	$\left(\frac{8}{45} - \frac{13\sqrt{3}}{160}\right)$	$\left(\frac{11}{45} - \frac{17\sqrt{3}}{120}\right)$	$\left(\frac{8}{45} - \frac{49\sqrt{3}}{480}\right)$	$-\left(\frac{21}{320} - \frac{3\sqrt{3}}{80}\right)$
$\frac{1}{2}$	$-\frac{7}{40}$	$\left(\frac{8}{45} + \frac{\sqrt{3}}{6}\right)$	$\frac{11}{45}$	$\left(\frac{8}{45} - \frac{\sqrt{3}}{6}\right)$	$\frac{3}{40}$
$\frac{1}{2} + \frac{\sqrt{3}}{4}$	$-\left(\frac{11}{320} + \frac{3\sqrt{3}}{80}\right)$	$\left(\frac{8}{45} + \frac{49\sqrt{3}}{480}\right)$	$\left(\frac{11}{45} + \frac{17\sqrt{3}}{120}\right)$	$\left(\frac{8}{45} + \frac{13\sqrt{3}}{160}\right)$	$-\left(\frac{21}{320} + \frac{3\sqrt{3}}{80}\right)$
1	$-\frac{1}{10}$	$\frac{16}{45}$	$\frac{22}{45}$	$\frac{16}{45}$	$-\frac{1}{10}$

Gen. Solution

$$y_{n+1} = y_n + h \sum_{i=1}^{5} b_i f_i$$

$$F_i = f(x + c_{ih}, y_i), y_i = y_n + h \sum_{j=1}^{5} a_{ij} F_i$$

See (8)

III. Methodology

We consider the font order initial value problem of ordinary differential equation.

$$y^1 = f(x, y), y(x_0) = y_0, \quad a \le x \le b. \qquad -(2.01)$$

We use a polynomial of the form

$$y(x) = \sum_{j=0}^{t-1} \propto_j (x) y_n + h \sum_{j=1}^{m-1} B_j(x) f(c_j) \qquad - \quad (2.02)$$

where t are the number of interpolation points x_{n+j} $(j = 0, 1, \ldots t-1)$ m is the number of collocation points c_j $(j = 0, 1, \ldots m-1)$, $f(x, y)$ is continuously differentiable. The constants $\propto_j$, B_j are elements of (t+m) x (t+m) square matrix. The are selected so that high accurate approximation of the solution of (2.01) is obtained, h is a constant step size.

$The \ltimes_j (x), B_j (x)$in (2.02) can be represented by polynomial of the form.

$$\ltimes_j (x) = \sum_{j=0}^{t-1} \ltimes_{j,i+1} x^i , \qquad j = (0, 1, \dots t-1)$$

$$B_j(x) = \sum_{j=1} h\, B_{j,i+1}\, x^1, \; j = (0, 1, \dots m-1) \qquad 2.03$$

The coefficient $\ltimes_{j,i+1}$ $B_j, i+1$ are to be determined. Now substituting (2.03) into (2.02), we have

$$y(x) = \sum_{j=0}^{m+t-1} \left(\sum_{j=0}^{t-1} \ltimes_{j,i+1} y_{n+1} + h \sum_{j=0}^{m-1} h\, B_{j,i+1} f_{n+j} \right) x^i = \sum_{i=0}^{m+t-1} \leftthreetimes_j x^i - (2.04)$$

Where

$$\leftthreetimes_j = \left(\sum_{j=0}^{t-1} \ltimes_{j,\; i+1} y_{n+j} + h \sum_{j=0}^{m-1} B_{j,i+1} f_{n+j} \right), \quad \leftthreetimes_j \in R^j, j \in (0, 1 \dots t+m-1).$$

(2.04) can be exposed in matrix form as y(x) = $(y_n, y_{n+1}, \dots y_{n+t-1}, f_n, f_{n+1} \dots f_{n+m-1})\, C^T D$
Where

$$C = \begin{pmatrix} \ltimes_{0,}\, 1 \dots \dots \ltimes_{t-1,}\, 1 \;\; hB_{0,}\, 1, \dots . hB_{m-1,}1 \\ \ltimes_{0,}\, 2 \dots \dots \ltimes_{t-1,}\, 2 \;\; hB_{0,}\, 2, \dots . hB_{m-1,}2 \\ \ltimes_{0,}\, 3 \dots \dots \ltimes_{t-1,}\, 3 \;\; hB_{0,}\, 3, \dots . hB_{m-1,}3 \\ \ltimes_{0,t+m} \quad \ltimes_{t-1,\; t+m} \quad hB_{0,}\, t+m, \dots . hP_{m-1,} t+m \end{pmatrix} \quad - \quad (2.\ 05)$$

$$D = \begin{pmatrix} 1 & x_{n,} & x_n^2 \;\dots . & x_n^{t+m-1} & \\ 0 & 1 & 2\, x_{n+q_1} \;\dots . & (t+m-1) & x_{n+q_1}^{t+m-2} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 1 & 2^x_{n+q_{m-1}} \;\dots & (t+m-1) & x_{n+q_{m-1}}^{t+m-2} \end{pmatrix} \quad (2.06)$$

q_1 are the collocation points.

Theorem 1.0

Let I denote identity matrix of dimension $(t+m)$, matrices C and D are defined by (2.05) and (2.06) respectively, satify.

(i) DC = I

(ii) $y(x) = \sum_{i=0}^{m+t-1} \leftthreetimes_j x^i$ - (2.07)

Now we assume a power series solution of degree order 4 of the form.

$$y(x) = \sum_{j=0}^{4} \leftthreetimes_j x^i, \qquad y^i(x) = \sum_{j=0}^{4} \leftthreetimes_j\, i\, x^{i-1} \qquad (2.08)$$

Interpolate at $x_{n,}$ and collocate at $x = x_{n+qi}$, $i = 0, \dots 4$, yields system of simultaneous equation of the form.

$\leftthreetimes_0 + \leftthreetimes_1 x_n + \leftthreetimes_2 x_n^2 + \leftthreetimes_3 x_n^3 + \leftthreetimes_4 x_n^4 = y_n$

$0 \quad \lambda_1 + 2\lambda_2 \ x_{n+q_1} + \ldots . \ 4\lambda_4 x^3_{n+q_1} = f_n + q_1$ (2.09)

. . . .

$0 \quad \lambda_1 \quad 2\lambda_2 x_{n+q_4} + 3\lambda_3 \ 4\lambda_4 x^3_{n+q_4} = f_n + q_4$

Where λ_j are to be determined.

(2.09) can be rewritten in matrix form as

$$\begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 \\ 0 & 1 & 2x_{n+q_1} & 3x^2_{n+q_1} & 4x^3_{n+q_1} \\ 0 & 1 & 2x_{n+q_2} & 3x^2_{n+q_2} & 4x^3_{n+q_2} \\ 0 & 1 & 2x_{n+q_3} & 3x^2_{n+q_3} & 4x^3_{n+q_3} \\ 0 & 1 & 2x_{n+q_4} & 3x^2_{n+q_4} & 4x^3_{n+q_4} \end{pmatrix} \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} y_n \\ f_n + q_1 \\ f_n + q_2 \\ f_n + q_3 \\ f_n + q_4 \end{pmatrix} \quad (2.10)$$

i.e

DA = Y, where

$$D = \begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 \\ 0 & 1 & 2x_{n+q_1} & 3x^2_{n+q_1} & 4x^3_{n+q_1} \\ . & . & . & . & . \\ 0 & 1 & 2x_{n+q_4} & 3x^2_{n+q_4} & 4x^3_{n+q_4} \end{pmatrix}$$

$$A = \left(\lambda_1 \quad \lambda_2 \quad \lambda_3 \quad \lambda_4 \ T\right) \qquad Y = y_n, \left(f_{n+q_1}, \ f_{n+q_2}, \ f_{n+q_3}, \ f_{n+q_4}, \right)^T \quad (2.11)$$

Using maple mathematical software we obtain the continuous scheme of the form.

i.e $\quad y(x) = y_n + h\left(f_{n+q_1} + f_{n+q_2} + f_{n+q_3} + f_{n+q_4}\right)$

$$y(x) = y_n + \left[\frac{-\,q_1 q_2 q_4 x}{(-q_4+q_1)(-q_2+q_1)(q_3+q_1)} + \frac{(h^2 q_2 q_4 + h^2 q_4 q_3 + h^2 q_1 q_3 + h^2 q_4 q_3)\, x^2}{2h^3(-q_3+q_1)(-q_2+q_1)(q_3+q_1)} - \frac{(q_2 h + q_3 h + q_4 h) x^3}{3\,h^3(-q_3+q_1)(-q_2+q_1)(-q_3+q_1)} + \frac{x^4}{4h^4(-q_4+q_1)(-q_2+q_1)(-q_3+q_1)} \right] f_n + q_1$$

$$+ \left[\frac{q_1 q_3 q_4 x}{(q_2-q_4)(-q_2+q_1)(q_2-q_3)} - \frac{(h^2_{q3} q_4 + h^2_{q1} q_4 + h^2_{q1} q_3) x^2}{2h^3(q_2-q_3)(-q_2+q_1)(q_2-q_4)} \right.$$

$$+\frac{(q_1h+q_4h+q_3h)x^3}{3\,h^3(q_1-q_3)(-q_2+q_1)(q_2-q_3} - \frac{x^4}{4\,h^3(q_2-q_4)(-q_2+q_1)(q_2-q_3)}\Bigg]$$
f_n+q_2

$$+\Bigg[\frac{q_1q_3q_4x}{(q_3-q_4)(q_1q_2-q_1q_3+q_3^2-q_1q_3)} + \frac{(h_{q1}^2q_4+h_{q1}^2q_2+h_{q2}^2q_4)\,x^2}{2\,h^3(q_1q_2-q_1q_3+q_3^2-q_2q_3)(q_3-q_4)}$$
$$-\frac{(q_2h+q_4h+q_1h)x^3}{3\,h^3(q_3-q_4)(q_1q_2-q_1q_3+q_3^2-q_2q_3)} + \frac{x^4}{4\,h^3(q_3-q_4)(q_1q_2-q_1q_3+q_3^2-}\Bigg]$$
$q2q3\,fn+q3$

$$+\Bigg[\frac{q_1q_3q_4x}{q_1q_2q_3-q_1q_2q_4+q_4^2q_1-q_1q_3q_4+q_4^2q_2-q_2q_3q_4-q_4^3+q_4^2q_3} -$$

$$\frac{(h^2q_1q_2+h^2q_1q_3+h^2q_2q_3)x^2}{2h^3(q_1q_2q_3-q_1q_2q_4+q_4^2q_1-q_1q_3q_4+q_4^2q_2-q_2q_3q_4-q_4^3+q_4^2q_3)} +$$

$$\frac{(q_1h+q_2h+q_3h)x^3}{3h^3(q_1q_2q_3-q_1q_2q_4+q_4^2q_1-q_1q_3q_4+q_4^2q_2-q_2q_3q_4-q_4^3+q_4^2q_3)} - \frac{x^4}{4h^3(q_1q_2q_3-q_1q_2q_4+q_4^2q_1-q_1q_3q_4+q_4^2q_2-q_2q_3q_4-q_4^3+q_4^2q_3)}\Bigg]\quad f_n+q_4$$

(2.12)

Now evaluating the continuous at $q_1=\left(\frac{1}{2}-\frac{3\sqrt{206}}{100}\right), q_2=\frac{33}{100},\ q_3=\frac{67}{100}, q_4=\left(\frac{1}{2}+\frac{3\sqrt{206}}{100}\right)$,
We obtain discrete schemes:

$$y_{n+q_1} = y_n + \left(\frac{1633}{18780} - \frac{71\sqrt{206}}{96717000}\right)hf_{n+q_1} + \left(\frac{134689}{939000} - \frac{927\sqrt{206}}{78250}\right)h\,f_{n+q_2} + \left(\frac{171511}{939000} - \frac{927\sqrt{206}}{78250}\right)hf_{n+q_3} + \left(\frac{1633}{18780} - \frac{121979\sqrt{206}}{19343400}\right)hf_{n+q_4}$$

$$y_{n+q_2} = y_n + \left(\frac{7623}{78250} + \frac{1629507\sqrt{206}}{257912000}\right)hf_{n+q_1} + \left(\frac{3470313}{21284000}hf_{n+q_2} - \frac{118701}{4256800}\right)hf_{n+q_3} + \left(\frac{7623}{78250} - \frac{1629507\sqrt{206}}{257912000}\right)hf_{n+q_4}$$

$$y_{n+q_3} = y_n + \left(\frac{8978}{117375} + \frac{1629507\sqrt{206}}{257912000}\right)hf_{n+q_1} + \frac{4520423}{12770400}hf_{n+q_2} + \frac{10410661}{63852000}hf_{n+q_3} + \left(\frac{8978}{117375} - \frac{1629507\sqrt{206}}{257912000}\right)hf_{n+q_4}$$

$$y_{n+q_4} = y_n + \left(\frac{1633}{18780} + \frac{121979\sqrt{206}}{19343400}\right)hf_{n+q_1} + \left(\frac{134689}{939000} + \frac{927\sqrt{206}}{78250}\right)hf_{n+q_2}$$

$$+\left(\frac{171511}{939000}-\frac{927\sqrt{206}}{78250}\right)hf_{n+q_3}+\left(\frac{1633}{18780}+\frac{17\sqrt{206}}{96717000}\right)hf_{n+q_4} \qquad -\ (2.13)$$

To convert to Runge-Kutta function evaluations, the discrete schemes (2.13) must satify (2.01), hence

$$y'_{n+q_1}=f(x_{n+q_1},y_{n+q_1})=f(x_{n+q_1},y_n+\left(\frac{1633}{18780}-\frac{71\sqrt{206}}{96717000}\right)hf_{n+q_1}$$
$$+\left(\frac{134689}{939000}-\frac{927\sqrt{206}}{78250}\right)hf_{n+q_2}+\left(\frac{171511}{939000}-\frac{927\sqrt{206}}{78250}\right)hf_{n+q_3}$$
$$+\left(\frac{1633}{18780}-\frac{121979\sqrt{206}}{19343400}\right)hf_{n+q_4}$$
$$y'_{n+q_2}=f(x_{n+q_2},y_{n+q_2})$$
$$=f(x_{n+q_2},y_n+\left(\frac{7623}{78250}+\frac{1629507\sqrt{206}}{257912000}\right)hf_{n+q_1}+\frac{3470313}{21284000}hf_{n+q_2}$$
$$-\frac{118701}{4256800}hf_{n+q_3}+\left(\frac{7623}{78250}-\frac{1629507\sqrt{206}}{257912000}\right)hf_{n+q_4}]$$
$$y'_{n+q_3}=f(x_{n+q_3},y_{n+q_3})=$$
$$f(x_{n+q_3},y_n+\left(\frac{8978}{117375}+\frac{1629507\sqrt{206}}{257912000}\right)hf_{n+q_1}+\frac{4520423}{12770400}hf_{n+q_2} \qquad (2.14)$$
$$+\frac{10410661}{63852000}hf_{n+q_3}+\left(\frac{8978}{117375}-\frac{1629507\sqrt{206}}{257912000}\right)hf_{n+q_4}]$$
$$y'_{n+q_4}=f(x_{n+q_4},y_{n+q_4})$$
$$=f(x_{n+q_4},y_n+\left(\frac{1633}{18780}+\frac{121979\sqrt{206}}{19343400}\right)hf_{n+q_1}+\left(\frac{927\sqrt{206}}{78250}\right)hf_{n+q_2}$$
$$+\left(\frac{171511}{939000}+\frac{927\sqrt{206}}{78250}\right)hf_{n+q_3}+\left(\frac{1633}{18780}+\frac{71\sqrt{206}}{96717000}\right)hf_{n+q_4}]$$

Putting $y'_{n+q_1}=K_1=f(x_{n+q_1},y_{n+q_1})or\ f_{n+q_1},K_2=f(x_{n+q_2},y_{n+q_2})or\ f_{n+q_2}\ etc.,$
We obtain the function evaluations as

$$K_1=f[x_{n+}\left(\frac{1}{2}-\frac{3\sqrt{206}}{100}\right)h,\ \ f_n+\left(\frac{1633}{18780}-\frac{71\sqrt{206}}{96717000}\right)hk_1$$
$$+\left(\frac{134689}{939000}-\frac{927\sqrt{206}}{78250}\right)hk_2$$
$$+\left(\frac{171511}{939000}-\frac{927\sqrt{206}}{78250}\right)hk_3+\left(\frac{1633}{18780}-\frac{121979\sqrt{206}}{19343400}\right)hk_4]$$
$$K_2=f[x_n+\frac{33}{100}h,\ \ y_n+\left(\frac{7623}{78250}+\frac{1629507\sqrt{206}}{257912000}\right)hk_1+h\left(\frac{3470313}{21284000}\right)hk_2$$
$$-\left(\frac{118701}{4256800}\right)hk_3$$
$$+\left(\frac{7623}{78250}-\frac{1629507\sqrt{206}}{257912000}\right)hk_4]$$
$$K_3=f[x_n+\frac{67}{100}h,\ \ y_n+\left(\frac{8978}{117375}+\frac{1629507\sqrt{206}}{257912000}\right)hk_1+\left(\frac{4520423}{12770400}\right)hk_2+\frac{10410661}{63852000}hk_3 \qquad (2.15)$$

$$+\left(\frac{8978}{117375}-\frac{1629507\sqrt{206}}{257912000}\right)hk_4]$$

$$K_4=f[x_{n+}\left(\frac{1}{2}+\frac{3\sqrt{206}}{100}\right)h,\ y_n+\left(\frac{1633}{18780}+\frac{121979\sqrt{206}}{19343400}\right)hk_1$$
$$+\left(\frac{134689}{939000}+\frac{927\sqrt{206}}{78250}\right)hk_2$$
$$+\left(\frac{171511}{939000}+\frac{927\sqrt{206}}{78250}\right)hk_3+\left(\frac{1633}{18780}+\frac{71\sqrt{206}}{96717000}\right)hk_4]$$

The weight is b = (b_1, b_2, b_3, b_4) ; Evaluating the continuous scheme (2.12) at $x =$

1 *we obtain*

$$b=\left(\frac{1633}{9390},\ \frac{1531}{4695},\ \frac{1531}{4695},\ \frac{1633}{9390}\right) \tag{2.16}$$

The four stage Runge-Kutta method is defined as

$$y_{n+1}=y_n+h\sum_{i=1}^{4}b_i\,k_i=y_n+\frac{1633}{9390}h\,(K_1+K_4)+\frac{1531}{4695}h\,(K_2+K_3)$$

Where $K_i\ \ i=1,2,\dots 3$ *are given by* (2.15)

3.00 Analysis of the scheme.

We can be write (2.15) in Butchers tableau as:

C	$\boldsymbol{A=(a_{ij}),\ \ i,j=1,2,\ \dots 4.}$				
$\frac{1}{2}-\frac{3\sqrt{206}}{100}$	$\left(\frac{1633}{18780}-\frac{71\sqrt{206}}{96717000}\right)$	$\left(\frac{134689}{939000}-\frac{927\sqrt{206}}{78250}\right)$	$\left(\frac{171511}{939000}-\frac{927\sqrt{206}}{78250}\right)$	$\left(\frac{1633}{18780}-\frac{121979\sqrt{206}}{19343400}\right)$	
$\frac{33}{100}$	$\left(\frac{7623}{78250}-\frac{1629507\sqrt{206}}{257912000}\right)$	$\frac{347013}{21284000}$	$-\frac{118701}{4256800}$	$\left(\frac{7623}{78250}+\frac{1629507\sqrt{206}}{257912000}\right)$	(2.16)
$\frac{67}{100}$	$\left(\frac{8978}{117375}+\frac{1629507\sqrt{206}}{257912000}\right)$	$\frac{4520423}{12770400}$	$\frac{10410661}{63852000}$	$\left(\frac{8978}{117375}+\frac{1629507\sqrt{206}}{257912000}\right)$	
$\frac{1}{2}+\frac{3\sqrt{206}}{00}$	$\left(\frac{1633}{18780}+\frac{121979\sqrt{206}}{19343400}\right)$	$\left(\frac{134689}{939000}+\frac{927\sqrt{206}}{78250}\right)$	$\left(\frac{171511}{939000}+\frac{927\sqrt{206}}{78250}\right)$	$\left(\frac{1633}{18780}+\frac{71\sqrt{206}}{96717000}\right)$	
b	$\frac{1633}{9390}$	$\frac{1531}{4695}$	$\frac{1531}{4695}$	$\frac{1633}{9390}$	

Or can be summarized as

C	A	*U*
	B	V

Where $(C=C_1C_2\ \dots\ C_4),^T\quad A=(a_{ij})\ \ y=1,2,\dots 4,\qquad U=(1,1,1,1),^T\ \ V=(1)$

(i) Consistency:

Te Runge-Kutta method is consistent since

$$\sum_{j=a_{ij}}^{4} a_{ij} = C_i, \quad \sum_{j=1}^{4} b_i = 1. \quad (\,see\ Table\ 2.16)$$

(ii) Stability:

The stability of the method is investigated by considering the linear test equation.

$y^{/} = \lambda y, \qquad \lambda \in \not\subset$

Putting $Z = \lambda$ h, $\quad h \in (0,1)$

The stability function is R (Z)

R (Z) = I + Zb^T(I − ZA)−1

where I is the identity matrix

$b = b_1,$

$b_2, \quad b_3, \quad b_4), the\ weight, e = (\,1,1,1,1), A\ is\ the\ Runge - Kutta\ matrix\ of$

coefficients (2.16).

The region of A- stability is the set of point satisfying R (Z) = { Z: R (Z)

$\leq |0|\ and$ Re Z $\leq 1\}$

The characteristic polynomial is defined as

P (Z) = det (R(Z) – I λ) , which can be potted using maple or math lap, to determine the region of A- stability.

(iii) Order and error of the scheme:

The order of the method is p = 2S = 8, since the nodes or absyssaes are Gaussian points of fourth order Gauss – Legendre Polynomial and all Gauss- quadrantine methods have order P = 2S, (S = stages) [3]. The Runge – Kutta solution y_{n+1} can be expanded into Taylors series as

$$y_{n+1} = y_n\ h \sum_{i=1}^{4} b_i K_i = y_n + h \sum_{i=1}^{4} b_i\ f\left(x_{n+q_1}, y_n + q_1\right)$$

$$= y_n + h \sum_{i=1}^{4} b_i \, y'_{n+q_1}$$

$$= y_n + C_1 h y'_n + C_2 h^2 y''_n + \ldots \quad C\, h^p y_{n(p+1)}$$

The R.K solution agrees with taylor's series expansion up to term in h^8. The truncation error is $0(h^9)$.

Numerical Experiments / Results:

To show the efficiency of this new method, we use three problems to compare our computational solutions with exact and other similar methods.

Problem I. (Electric circuit problem model)

In an Electric RL – circuit system, L is the inductance constant, I the current, R the resistance constant, and E9t), periodic electronmotive force. Find the currents at time t = 0.2 (0.2) 0.1 secondly; with L = .1, R 5, E(t) = sin(t), with initial condition I (0) = 1.0.

Model: By Kirchlotts voltage law (KVL) the voltage across the resister is RI, the voltage drop across inductor is E_L. By KVL Law the sum of two voltage must be equal to electromotive force E(t).

Thus our problem reduces to differential equation $L\frac{dI}{dt} + RI = E(t)$

i.e. $\frac{dI}{dt} = -50\, I + 10 \sin(t)\,, \quad y(0) = 1.0.$

Example 2: (Logistic Population Model)

A logistic mathematic model has useful application to human population and animal populations [ ]. The mathematical model is defined as

$y' = Ay(t) - B\, y^2(t), \qquad y(t_0) = y_{01} \quad A, \quad B \quad > 0.$

If $y(0) < \frac{A}{B}$, the population is monotonic increasing to limit $\frac{A}{B}$ and if $y(0) > \frac{A}{B}$, it decreases to limit $\frac{A}{B}$.

Now we consider a logistic problem

$y' = Ay(t) - B\, y^2(t),\ y(t_0) = y_0 \quad with\ A = 5 \quad B = 3$ initial condition

$y(0) = 2$ Million Cows in Kaduna State, Nigeria, t is measured in months.

Model: The logistic problem reduces to differential equation

$y' = 5y - 3y^2, \qquad y(0) = 2, \quad h = .2$

Example 3: (Chemical Reaction Model):

A bimolecular reactions a and B combines a$\in A$ moles per litre of distance $A\ and\ b \in B$ moles of substance B given y(t) is the number of moles per litre that have reacted after time t we can find the number of moles per litre which reacted at given times.

Model: We use chemical reaction Law) which states that under constant temperature, the chemical reaction is proportional to the product of concentrations of the reacting substances, thus the model can be defined by a differential equation:

$y' = K(a-y)(b-y), \quad y(t_0) = y_0$, K is the proportional constant, a, b are constants.

In our example we consider $a = 5,\ b = 3, y(0) = 0$ *and* $K = 1$. Thus our model reduce to an initial value problem $\quad y' = y^2 - 8y + 15, \quad y(0) = 0.$

We can compute the solutions at $t = 0\ (.1)\ .5$.

The following are the solution tables for comparison.

Notations:

$y(x_{n+1}), n = 0, 1, \ldots =$ Exact solutions, with step – size h.

y_{n+1} = The new R-.K approximate solutions.

Abs = $/y(_{n+1}) - y_{n+1}/$, *absolute error*.

ER_1 = absolute error of – point Gauss – quadrature (3)method.

ER_2 = absolute error of Lie et al (8)method

Table 1: Comparison of numerical solution of problem I. (RI-Circuit).

Analytic solution, $y(t) = \frac{500}{2501}\sin(t) + \frac{2511}{2501}\ e^{-50t} - \frac{10}{2501}\cos(t)$

t	$y(t)$	y_{n+1}	ER_1	ER_2	Abs
.02	0.3693509037	0.3693509135	3.81 E-06	3.81 E-06	9.87 E-09
.04	0.1398758745	0.1398758818	2.80 E-06	1.57 E-06	7.27 E-09
.06	0.0579829374	0.0579829414	1.55 E-06	8.65 E-07	4.01 E-09
.08	0.0303798097	0.0303798097	7.58 E-07	4.24 E-07	1.97 E-09
.10	0.2274516347	0.2274516347	3.49 E-07	1.95 E-07	9.04-10

Table 2: Comparison of Numerical solution of problem 2.

t	$y(t)$	y_{n+1}	ER_1	ER_2	Abs
.2	1.7755301747	1.7755301873	3.37 E-06	2.13 E-06	1.25 E-08
.4	1.7051273147	1.7051273208	1.74 E-06	1.57 E-06	6.12 E-09
.6	1.6806121251	1.6806121278	7.92 E-07	8.65 E-07	2.66 E-09
.8	1.6717699224	1.6717699235	3.44 E-07	4.24 E-07	1.12 E-09
.10	1.6685404228	1.6685404233	1.46 E-07	1.95 E-07	4.62 E-10

Analytic Solution: $\boldsymbol{y(t) = -\frac{10}{-6+e^{-st}}}$

Table 3: Comparison solution of problem 3.

Analytic Solution: $y(t) = \frac{15(e^{2t}-1)}{5e^{-2t}-3}$

t	$y(t)$	y_{n+1}	ER_1	ER_2	Abs
.1	1.0688853015	1.0688853144	9.93 E-08	5.18 E-06	1.30 E-08
.2	1.6544440817	1.6544440919	7.72 E-08	3.66 E-06	1.02 E-08
.3	2.0180987318	2.0180987391	5.55 E-08	2.49 E-06	7.31 E-09
.4	2.2617841981	2.2617842034	4.03 E-08	1.17 E-06	5.28 E-09
.5	2.4335031421s	2.4335031460	2.98 E-08	1.26 E-06	3.89 E-09

IV. Discussion / Conclusion

The new method has four stages compared with Lie and Norsett with five stages [8] thus the implementation cost is cheaper. The new method converges better and faster to the exact solutions (see comparison tables above), has biger area of A-stability than methods [3] and [8] consequently the have developed a new Gauss- Quadrature Runge-Kutta method with four points or stages only.

References

[1.] Bucher, J.C. 91996): **A history of Runge Kutta methods apply, Numerical Mathematics**, P247-260

[2.] Buctcher J.C. (2013) **Numerical methods for ordinary Differential Equations second** Edition. John Wiley & Sons Ltd.

[3.] Golllub, Gene H; C & Welschi, John H. (1999) Calculation of Gauss Quadrature Rules. *Math. Computation 23 (106 P 221 -230*

[4.] Haizer E & Wanner G. (1991) **Solving Ordinary Differential Equations II, stiff and Differential algebraic problems**, Springer, berlin.

[5.] Kuntzmann J. (1961): Neure Entwickhengsweisen der methoden Von Runge und kutta, Z. Angew Math Mech. 41, T28-T31.

[6.] Lambert J.D (1973) **Computational Methods in Ordinary Differential Equations** John Wiley and Sons Ltd, New York.

[7.] Laurie, Duk P. (2001) Computational of Gauss –type quadrature formula Jorn. Of comp. applied math.

[8.] Lie, J and and Norsett S.P (1989) Super convergence for multistep collocation, math compt. (52) P 65-79.

[9.] Press, W.H, Teukolsky S.A; Vetterling W.T & Flannery B.P (2007). “Section 4.6, Gaussian Quadratures and orthogonal polynomials. New York Cambridge University Press.