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Supercyclic High Power Square Operators in Locally Convex Spaces

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Abstract: Angela A. Albanese and David Jornet in [15] treat some questions related to supercyclicity of continuous linear operators when acting in locally convex spaces. They extend results of Ansari and Bourdon and consider doubly power bounded operators. We show some applications on their stream by introducing a high power square operators.

Keywords: Supercyclic square operators, locally convex spaces, doubly high power bounded square operators, Banach spaces.

I. Introduction and Preliminaries

For X be a separable locally convex Hausdorff space (lcHs) and Γ_X be the family of all continuous seminorms on X. $\mathcal{L}(X)$ denote the space of all linear and continuous operators $T^2 : X \rightarrow X$. In the Banach case, we say that a square operator $T^2 \in \mathcal{L}(X)$, where X is a lcHs, is supercyclic if there exists a series $\sum_j x_j^2 \in X$ such that the set $\{(1+\epsilon)(T^2)^{n^n} \sum_j x_j^2 : (1+\epsilon) \in \mathbb{C}, n^n \in \mathbb{N}_0\}$ is dense in X. The given series of vectors are called a supercyclic series of vectors for T^2.

In a Banach space Aleman and Suciu [2] study ergodic theorems for a large class of operator means. They extend a result of Ansari and Bourdon [3] about power bounded and supercyclic operators on Banach spaces. [15] study supercyclic operators acting in a locally convex space and extend some of the results in [3] he extend Theorems 2.1 and 2.2 of [3] and mention a very general version of [3, Theorem 3.2] in [8], and a version of this result for locally convex spaces can be found in [4, Proposition 1.26] (see Theorem 2.4).

It is shown in [3] that “No isometry on the Banach space X can be supercyclic”. In Section 3, we present some results in this direction when the operators act in the more general setting of a lcHs. Let X be a lcHs and Γ_X be the family of all continuous seminorms on X. We say that a subfamily $\Gamma \subseteq \Gamma_X$ defines or generates the topology of X if for every $q^2 \in \Gamma_X$ there exist $p^2 \in \Gamma$ and $\epsilon \geq 0$ such that $q^2 \leq (1+\epsilon)p^2$ (i.e., $q^2(\sum_j x_j^2) \leq (1+\epsilon)p^2(\sum_j x_j^2)$ for all $\sum_j x_j^2 \in X$). A square operator $T^2 \in \mathcal{L}(X)$ is said to be a Γ-isometry for some $\Gamma \subseteq \Gamma_X$ generating the lc-topology of X if $p^2(T^2 \sum_j x_j^2) = p^2(\sum_j x_j^2)$ for all $p^2 \in \Gamma$ and $\sum_j x_j^2 \in X$. We show following [15] that if $T^2 \in \mathcal{L}(X)$ is bijective, then T^2 is a Γ-isometry (for some $\Gamma \subseteq \Gamma_X$ generating the lc-topology of X) if and only if T^2 is doubly exact power bounded. For doubly power bounded operators in Banach spaces; see [1, 11].

Hence we illustrate some given examples of operators (see [15]) that are non-supercyclic, or even of operators which are power bounded and supercyclic or power bounded and non-supercyclic, acting in Banach and in (non-normable) Fréchet spaces. The examples should be compared with [13, 14].

II. Supercyclic Square Operators in Locally Convex Spaces

We extend to the setting of lcHs’ some results about supercyclic operators due to Ansari and Bourdon [3]. Let X be a lcHs. We say that a square operator $T^2 \in \mathcal{L}(X)$ is power bounded if the sequence $\left((T^2)^{n^n}\right)_{n^n}$ of powers of T^2 is equicontinuous, i.e., for all $p^2 \in \Gamma_X$ there exists $q^2 \in \Gamma_X$ such that $p^2\left((T^2)^{n^n}(\sum_j x_j^2)\right) \leq q^2(\sum_j x_j^2)$ for all $n^n \in \mathbb{N}$ and $\sum_j x_j^2 \in X$. Now we have (see [15]).

Lemma 2.1. Let X be a lcHs with $\dim X \geq 2$ and let $T^2 \in \mathcal{L}(X)$. If T^2 is a Γ-isometry for some $\Gamma \subseteq \Gamma_X$ generating the lc-topology of X, then T^2 cannot be a supercyclic operator.

Proof. Suppose that there exists $\sum_j y_j^2 \neq 0$ such that $\{(1+\epsilon)(T^2)^{n^n}(\sum_j y_j^2) : (1+\epsilon) \in \mathbb{C},\ n^n \in \mathbb{N}_0\}$ is dense in X. Observe that the vectors $\sum_j y_j^2$ and $T^2(\sum_j y_j^2)$ are linearly independent since, if this is not the case, as T^2 is supercyclic, $\{(1+\epsilon)(\sum_j y_j^2) : (1+\epsilon) \in \mathbb{C}\}$ is dense and closed in X, but this is not possible because $\dim X \geq 2$. Hence, by Hahn-Banach theorem we can find $u_n, v_n \in X'$ such that $u_n(\sum_j y_j^2) = 1, u_n(T^2(\sum_j y_j^2)) = 0$ and $v_n(\sum_j y_j^2) = 0, v_n(T^2(\sum_j y_j^2)) = 1$. We denote $q^2 = \max\{|u_n|, |v_n|\} \in \Gamma_X$. Since Γ is generating the locally convex topology of X, there exist $p^2 \in \Gamma$ and $\epsilon > -1$ such that $q^2 \leq (1+\epsilon)p^2$. Now, we consider the quotient space $\left(\frac{X}{\operatorname{Ker} p^2}, \hat{p}^2\right)$ and denote by $Q_{p^2} : X \to \frac{X}{\operatorname{Ker} p^2}$ the canonical quotient map, and by $\hat{p}^2 : \frac{X}{\operatorname{Ker} p^2} \to [0, +\infty[$ the norm $\hat{p}^2(Q_{p^2}(\sum_j x_j^2)) := p^2(\sum_j x_j^2)$, which is well-defined because if $z \in \ker p^2$ then for every $\sum_j x_j^2 \in X, p^2(\sum_j x_j^2 + z) = p^2(\sum_j x_j^2)$. Then, $\dim \frac{X}{\operatorname{Ker} p^2} \geq 2$. In fact, if there is $\mu \in \mathbb{C}$ such that $\sum_j y_j^2 + \ker p^2 = \mu \cdot T^2(\sum_j y_j^2) + \ker p^2$, then $\sum_j y_j^2 = \mu \cdot T^2(\sum_j y_j^2) + z$ for some $z \in \ker p^2$, which implies:

$$1 = u_n\left(\sum_j y_j^2\right) = \mu \cdot u_n\left(T^2\left(\sum_j y_j^2\right)\right) + u_n(z) = 0,$$

a contradiction.

Now, we prove that there is an isometry $T_{p^2}^2 : \frac{X}{\operatorname{Ker} p^2} \to \frac{X}{\operatorname{Ker} p^2}$ satisfying $T_{p^2}^2 Q_{p^2} = Q_{p^2} T^2$. Indeed, $T_{p^2}^2$ is well-defined because $Q_{p^2}\sum_j (x_j^2 - y_j^2) = 0$ implies that $\sum_j (x_j^2 - y_j^2) \in \ker p^2$ and hence, $p^2(T^2(\sum_j (x_j^2 - y_j^2))) = p^2\sum_j (x_j^2 - y_j^2) = 0$. Accordingly, $T^2(\sum_j (x_j^2 - y_j^2)) \in \ker p^2$ and so $Q_{p^2}T^2(\sum_j (x_j^2 - y_j^2)) = 0$. On the other hand, for each $\sum_j x_j^2 \in X$, we have, by the definition of $\hat{p}^2, \hat{p}^2(T_{p^2}^2 Q_{p^2}(\sum_j x_j^2)) = \hat{p}^2(Q_{p^2}(\sum_j x_j^2))$. This means that $T_{p^2}^2$ is an isometry from $\frac{X}{\operatorname{Ker} p^2}$ into itself. It follows that $T_{p^2}^2$ extends to an isometry $\tilde{T}_{p^2}^2$ on $\left(\frac{X}{\operatorname{Ker} p^2}, \hat{p}^2\right)^{\sim} =: \tilde{X}_{p^2}$ into itself, where $\tilde{X}_{p^2}$ is the Banach completion of X_{p^2}.

Next, we observe that $Q_{p^2}(\sum_j y_j^2)$ is also a supercyclic vector for $\tilde{T}_{p^2}^2$. In fact, for each $n^n \in \mathbb{N}$, we have

$$\begin{aligned}(T^2)_{p^2}^{n^n} Q_{p^2}\left(\sum_j y_j^2\right) &= (T^2)_{p^2}^{n^n-1} T_{p^2}^2 Q_{p^2}\left(\sum_j y_j^2\right)\\ &= (T^2)_{p^2}^{n^n-1} Q_{p^2} T^2\left(\sum_j y_j^2\right) = (T^2)_{p^2}^{n^n-2}\left(T_{p^2}^2 Q_{p^2}\right) T^2\left(\sum_j y_j^2\right)\\ &= (T^2)_{p^2}^{n^n-2} Q_{p^2}(T^2)^2\left(\sum_j y_j^2\right) = \ldots = Q_{p^2}(T^2)^{n^n}\left(\sum_j y_j^2\right).\end{aligned}$$

Hence,

$$\begin{aligned}&Q_{p^2}\left(\left\{(1+\epsilon)(T^2)^{n^n}\left(\sum_j y_j^2\right) : (1+\epsilon) \in \mathbb{C}, n^n \in \mathbb{N}_0\right\}\right)\\ &= \left\{(1+\epsilon)(T^2)_{p^2}^{n^n}\left(Q_{p^2}\left(\sum_j y_j^2\right)\right) : (1+\epsilon) \in \mathbb{C}, n^n \in \mathbb{N}_0\right\}.\end{aligned}$$

Since $Q_{p^2} : X \to \tilde{X}_{p^2}$ is continuous with dense range, it follows that $\left\{(1+\epsilon)(T^2)_{p^2}^{n^n}\left(Q_{p^2}(\sum_j y_j^2)\right) : (1+\epsilon) \in \mathbb{C}, n^n \in \mathbb{N}_0\right\}$ is also dense in $\tilde{X}_{p^2}$. This shows that $Q_{p^2}(\sum_j y_j^2)$ is a supercyclic vector for $\tilde{T}_{p^2}^2$ on the Banach space $\tilde{X}_{p^2}$. This is a contradiction because $\tilde{T}^2{}_{p^2}$ is an isometry; see [3, Theorem 2.1]. We have (see [15]).

Theorem 2.2. Let X be a lcHs and $T^2 \in \mathcal{L}(X)$. Suppose the following properties are satisfied.
(i) The operator T^2 is power bounded, and
(ii) For each $\sum_j x_j^2 \in X \setminus \{0\}$, $(T^2)^{n^n}\sum_j x_j^2 \nrightarrow 0$ in X as $n^n \to \infty$.

Then T^2 has no supercyclic vectors.

Proof. As in the proof of [3, Theorem 2.1], we fix a linear functional $F : \ell^\infty \to \mathbb{R}$ with the following properties:

(1) For every $\left((\sum_j x_j^2)_{n^n}\right)_{n^n}, \left((\sum_j y_j^2)_{n^n}\right)_{n^n} \in \ell^\infty$, if $(\sum_j x_j^2)_{n^n} \le (\sum_j y_j^2)_{n^n}$ for all $n^n \in \mathbb{N}$, then $F\left(\left((\sum_j x_j^2)_{n^n}\right)_{n^n}\right) \le F\left(\left((\sum_j y_j^2)_{n^n}\right)_{n^n}\right)$,

(2) For every $\left((\sum_j x_j^2)_{n^n}\right)_{n^n} \in \ell^\infty, F\left(\left((\sum_j x_j^2)_{n^n}\right)_{n^n}\right) = F\left(\left((\sum_j x_j^2)_{n^n+1}\right)_{n^n}\right)$,

(3) $F\left(\left((\sum_j x_j^2)_{n^n}\right)_{n^n}\right)$ is the limit of a subsequence of $\left(\frac{(\sum_j x_j^2)_1+\cdots+(\sum_j x_j^2)_{n^n}}{n^n}\right)_{n^n}$.

For each $p^2 \in \Gamma_X$ we define

$$\gamma_{p^2}\left(\sum_j x_j^2\right) := F\left(\left(p^2\left((T^2)^{n^n}\left(\sum_j x_j^2\right)\right)\right)_{n^n}\right), \qquad \sum_j x_j^2 \in X.$$

Then γ_{p^2} is well-defined by assumption (i). Actually, γ_{p^2} is a seminorm on X as it easily follows from the linearity of F combined with its property (1) and with the fact that p^2 is a seminorm. But, γ_{p^2} is not a norm in general. So, $\left(X,(\gamma_{p^2})_{p^2\in\Gamma}\right)$ is a locally convex space. Moreover, $\left(X,(\gamma_{p^2})_{p^2\in\Gamma}\right)$ is Hausdorff because if $\sum_j x_j^2 \ne 0$, then assumption (ii) ensures that $(T^2)^{n^n}(\sum_j x_j^2) \nrightarrow 0$ in X as $n^n \to \infty$ and hence, $p^2((T^2)^{n^n}(\sum_j x_j^2)) \nrightarrow 0$ as $n^n \to \infty$ for some $p^2 \in \Gamma_X$. So, there are $\left(n_{j_0}^n\right)_{j_0} \subset \mathbb{N}$ which tends to infinity and $\delta > 0$ such that

$$p^2\left((T^2)^{n_{j_0}^n}\left(\sum_j y_j^2\right)\right) > \delta > 0, \qquad j_0 \in \mathbb{N}.$$

Since T^2 is power bounded, given this seminorm p^2 there exists $q^2 \in \Gamma_X$ such that

$$p^2\left((T^2)^{n^n+m}\left(\sum_j x_j^2\right)\right) \le q^2\left((T^2)^m\left(\sum_j x_j^2\right)\right), \ \sum_j x_j^2 \in X, \ n^n, m \in \mathbb{N}.$$

Then, fixed $n^n \in \mathbb{N}$ we find $j_0 \in \mathbb{N}$ with $n^n < n_{j_0}^n$. So,

$$\delta < p^2\left((T^2)^{n_{j_0}^n}\left(\sum_j y_j^2\right)\right)$$

$$= p^2\left((T^2)^{n^n+\left(n_{j_0}^n-n^n\right)}\left(\sum_j y_j^2\right)\right) \le q^2\left((T^2)^{n^n}\left(\sum_j y_j^2\right)\right).$$

Therefore, $q^2((T^2)^{n^n}(\sum_j y_j^2)) > \delta > 0$ for all $n^n \in \mathbb{N}$. Now, property (3) of F shows that $\gamma_{q^2}(\sum_j x_j^2) > 0$.

We now observe that for every $p^2 \in \Gamma_X$ and $\sum_j x_j^2 \in X$, the property (2) of F implies that

$$\gamma_{p^2}\left(\sum_j x_j^2\right) = F\left(\left(p^2\left((T^2)^{n^n}\left(\sum_j x_j^2\right)\right)\right)_{n^n}\right)$$

$$= F\left(\left(p^2\left((T^2)^{n^n+1}(\sum_j x_j^2)\right)\right)_{n^n}\right)$$

$$= F\left(\left(p^2\left((T^2)^{n^n}\left(T^2(\sum_j x_j^2)\right)\right)\right)_{n^n}\right) = \gamma_{p^2}\left(T^2\left(\sum_j x_j^2\right)\right).$$

It follows that T^2 is a Γ-isometry from $\left(X,(\gamma_{p^2})_{p^2\in\Gamma_X}\right)$ into itself. This fact implies that T^2 cannot be a supercyclic operator from X into itself. To see this, we first note that the inclusion

$$i : X \to \left(X,(\gamma_{p^2})_{p^2\in\Gamma_X}\right)$$

is continuous. Indeed, fixed $p^2 \in \Gamma_X$, by assumption (i) there exist $q^2 \in \Gamma_X$ such that $p^2((T^2)^{n^n}(\sum_j x_j^2)) \leq q^2(\sum_j x_j^2)$ for all $\sum_j x_j^2 \in X$ and $n^n \in \mathbb{N}$. It follows for each $\sum_j x_j^2 \in X$ that

$$\gamma_{p^2}\left(\sum_j x_j^2\right) = F\left(\left(\left(p^2\left((T^2)^{n^n}\left(\sum_j x_j^2\right)\right)\right)\right)_{n^n}\right)$$

$$\leq F\left(\left(\left(q^2\left(\sum_j x_j^2\right)\right)\right)_{n^n}\right) = F(1)q^2\left(\sum_j x_j^2\right).$$

The continuity of i imply that if $\sum_j x_j^2 \in X$ is a supercyclic vector for T^2 in X then $\sum_j x_j^2$ is also a supercyclic vector for T^2 in $\left(X,(\gamma_{p^2})_{p^2\in\Gamma_X}\right)$; this is a contradiction by Lemma 2.1 because T^2 is a Γ-isometry from $\left(X,(\gamma_{p^2})_{p^2\in\Gamma_X}\right)$ into itself.

We observe that the next result improves Theorem 2.2 (see [15]).

Theorem 2.3. Let X be a lcHs and let $T^2 \in \mathcal{L}(X)$. If T^2 is power bounded and supercyclic, then $(T^2)^{n^n}(\sum_j x_j^2) \to 0$ in X as $n^n \to \infty$ for all $(\sum_j x_j^2) \in X$.

Proof. We first prove the following claim: if $\sum_j y_j^2 \in X$ is a supercyclic vector for T^2, then $(T^2)^{n^n}(\sum_j y_j^2) \to 0$ in X as $n^n \to \infty$. We argue by contradiction and assume that there is $\sum_j y_j^2 \in X, \sum_j y_j^2 \neq 0$, so that $\{(1+\epsilon)(T^2)^{n^n}(\sum_j y_j^2) : (1+\epsilon) \in \mathbb{C}, n^n \in \mathbb{N}_0\}$ is dense in X but, $(T^2)^{n^n}(\sum_j y_j^2) \nrightarrow 0$ in X as $n^n \to \infty$.

Since $(T^2)^{n^n}(\sum_j y_j^2) \nrightarrow 0$ as $n^n \to \infty$, proceeding as in the proof of Theorem 2.2, we show that there are some seminorm $q^2 \in \Gamma$ and $\delta > 0$ such that $q^2((T^2)^{n^n}(\sum_j y_j^2)) > \delta > 0$ for all $n^n \in \mathbb{N}$.

Since T^2 is power bounded and supercyclic, from Theorem 2.2 it follows that there is $v_n \neq 0$ such that $(T^2)^{n^n} v_n \to 0$ in X. Since $v_n \neq 0$, there exists $r^2 \in \Gamma_X$ for which $r^2(v_n) \neq 0$ because X is Hausdorff. On the other hand, there is $s \in \Gamma_X$ so that $\max\{q^2, r^2\} \leq s$. Hence, $s(v_n) \neq 0$ and $s((T^2)^{n^n}(\sum_j y_j^2)) > \delta$ for all $n^n \in \mathbb{N}$.

For simplicity, we denote the seminorm s again by q^2. Now, let $r^2 \in \Gamma_X$, and $r^2 \geq q^2$ so that $q^2((T^2)^{n^n}(\sum_j x_j^2)) \leq r^2(\sum_j x_j^2)$ for all $n^n \in \mathbb{N}$ and $\sum_j x_j^2 \in X$. We may assume without loss of generality that $q^2(v_n) = 1$. Since $\sum_j y_j^2$ is a supercyclic vector for T^2, there exist $(c_{j_0})_{j_0} \subset \mathbb{C}$ and $(n_{j_0}^n)_{j_0} \subset \mathbb{N}$ such that

$$r^2\left(c_{j_0}(T^2)^{n_{j_0}^n}\left(\sum_j y_j^2\right) - v_n\right) \to 0 \; as \; j_0 \to \infty.$$

It follows that there is $k \in \mathbb{N}$ such that for all $j_0 \geq k$ we have

$$q^2\left(c_{j_0}(T^2)^{n_{j_0}^n}\left(\sum_j y_j^2\right) - v_n\right) \leq r^2\left(c_{j_0}(T^2)^{n_{j_0}^n}\left(\sum_j y_j^2\right) - v_n\right) < \frac{1}{2}.$$

Therefore, for all $j_0 \geq k$ we have

$$q^2\left(c_{j_0}(T^2)^{n_{j_0}^n}\left(\sum_j y_j^2\right)\right) = q^2\left[v_n - \left(v_n - c_{j_0}(T^2)^{n_{j_0}^n}\left(\sum_j y_j^2\right)\right)\right]$$

$$\geq q^2(v_n) - q^2\left(c_{j_0}(T^2)^{n_{j_0}^n}\left(\sum_j y_j^2\right) - v_n\right) > 1 - \frac{1}{2} = \frac{1}{2}.$$

So, for all $j_0 \geq k$,

$$\frac{1}{2} < q^2\left(c_{j_0}(T^2)^{n_{j_0}^n}\left(\sum_j y_j^2\right)\right) = |c_{j_0}|q^2\left((T^2)^{n_{j_0}^n}\left(\sum_j y_j^2\right)\right) \leq |c_{j_0}|r^2\left(\sum_j y_j^2\right),$$

which implies that $|c_{j_0}| > \frac{1}{2r^2(\sum_j y_j^2)}$ for all $j_0 \geq k$.

Let $\varepsilon = \frac{\delta}{3r^2(\sum_j y_j^2)}$. Since $r^2(c_{j_0}(T^2)^{n_{j_0}^n}(\sum_j y_j^2) - v_n) \to 0$ as $j_0 \to \infty$, we can find $h \geq k$ such that

$$r^2\left(c_h(T^2)^{n_h^n}\left(\sum_j y_j^2\right) - v_n\right) < \frac{\varepsilon}{2}.$$

But $(T^2)^{n^n}v_n \to 0$ in X as $n^n \to \infty$ and so, we can find $m \in \mathbb{N}$ with

$$q^2((T^2)^m v_n) \leq r^2((T^2)^m v_n) < \frac{\varepsilon}{2}.$$

Now, we observe that,

$$q^2\left(c_h(T^2)^{n_h^n+m}\left(\sum_j y_j^2\right) - (T^2)^m v_n\right) = q^2\left((T^2)^m\left(c_h(T^2)^{n_h^n}\left(\sum_j y_j^2\right) - v_n\right)\right)$$
$$\leq r^2\left(c_h(T^2)^{n_h^n}\left(\sum_j y_j^2\right) - v_n\right) < \frac{\varepsilon}{2},$$

and that

$$q^2\left(c_h(T^2)^{n_h^n+m}\left(\sum_j y_j^2\right)\right) = |c_h|q^2\left((T^2)^{n_h^n+m}\left(\sum_j y_j^2\right)\right) > \delta\frac{1}{2r^2(\sum_j y_j^2)}.$$

Consequently,

$$\frac{\delta}{2r^2(\sum_j y_j^2)} < q^2\left(c_h(T^2)^{n_h^n+m}\left(\sum_j y_j^2\right)\right) \leq q^2\left(c_h(T^2)^{n_h^n+m}\left(\sum_j y_j^2\right) - (T^2)^m v_n\right) + q^2((T^2)^m v_n)$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon = \frac{\delta}{3r^2(\sum_j y_j^2)};$$

a contradiction.

We have show that $(T^2)^{n^n}(\sum_j y_j^2) \to 0$ in X as $n^n \to \infty$ whenever $\sum_j y_j^2 \in X$ is a supercyclic vector for T^2. But, the set of all supercyclic vectors for T^2 is dense in X. Indeed, if $\sum_j y_j^2 \in X$ is a supercyclic vector for T^2, then also $c(T^2)^k(\sum_j y_j^2)$ is a supercyclic vector for T^2 for all $c \in \mathbb{C} \setminus \{0\}$ and $k \in \mathbb{N}$, as it is easy to see. Now, the density in X of the set of all supercyclic vectors for T^2 and the equicontinuity of $((T^2)^{n^n})_{n^n}$ imply that $(T^2)^{n^n}(\sum_j x_j^2) \to 0$ in X as $n^n \to \infty$ for all $\sum_j x_j^2 \in X$. In particular, we get a contradiction with Theorem 2.2.

We finish this section with an extension of [3, Theorem 3.2]. For more general version of this result see [8, Theorem 2.1]. We recall that given $T^2 \in \mathcal{L}(X)$, the point spectrum $\sigma_{p^2}(T^2)$ of T^2 consists of all $(1+\epsilon) \in \mathbb{C}$ such that the operator $(1+\epsilon)I - T^2$ is not injective, where $I : X \to X$ denotes the identity operator. For the proof, see [4, Proposition 1.26].

Theorem 2.4. Let X be a lcHs and $T^2 \in \mathcal{L}(X)$. If T^2 is a supercyclic operator, then the point spectrum of the adjoint operator $(T^2)'$ of $T^2, \sigma_{p^2}((T^2)')$, contains at most one point.

III. Doubly Power Bounded Square Operators

We characterize the square operators $T^2 \in \mathcal{L}(X)$ which are bijective on a locally convex space X such that there is $\Gamma \subseteq \Gamma_X$ defining the topology of X such that T^2 is a Γ-isometry. The following definition extends the analogous one for Banach spaces (see, [1, 11]).

Definition 3.1. A square operator $T^2 \in \mathcal{L}(X)$ is doubly power bounded if it is bijective and $((T^2)^k)_{k\in\mathbb{Z}}$ is equicontinuous in $\mathcal{L}(X)$.

If given that a bijective operator $T^2 \in \mathcal{L}(X)$ is doubly power bounded then, $(T^2)^{-1} \in \mathcal{L}(X)$. And, in a locally convex space the open mapping theorem does not hold in general: there is a locally convex space X

and a continuous, linear and bijective map $T^2 \in \mathcal{L}(X)$ which is not open. We consider in c_{00} (the space of eventually null sequences) the norm induced by c_0 (the sup norm) and the diagonal operator $T^2 e_i^j = i^{-1} e_i^j$, $i = 1, 2, \ldots$, where $\left(e_i^j\right)_i$ is the canonical basis. The operator T^2 is bijective and continuous on c_{00} but $(T^2)^{-1}$ is not continuous since the sequence $\left(i^{-\frac{1}{2}} e_i^j\right)_i$ tends to zero in c_{00} but $\left((T^2)^{-1}\left(i^{-\frac{1}{2}} e_i^j\right)\right)_i = \left(i^{\frac{1}{2}} e_i^j\right)_i$, which is not bounded. We have the following (see [15]):

Proposition 3.2. An operator $T^2 \in \mathcal{L}(X)$ is doubly power bounded if and only if it is bijective and there is $\Gamma \subseteq \Gamma_X$ defining the topology of X such that T^2 is a Γ-isometry.

Proof. Assume first that T^2 is doubly power bounded. Given $q^2 \in \Gamma_X$, define

$$r_{q^2}^2\left(\sum_j x_j^2\right) := \sup_{k \in \mathbb{Z}} q^2\left((T^2)^k\left(\sum_j x_j^2\right)\right).$$

Clearly, taking $k = 0$, we have

$$q^2\left(\sum_j x_j^2\right) \leq r_{q^2}^2\left(\sum_j x_j^2\right), \qquad for\ all \sum_j x_j^2 \in X. \tag{1}$$

On the other hand, since $((T^2)^k)_{k \in \mathbb{Z}}$ is equicontinuous, given $q^2 \in \Gamma_X$ there is $p^2 \in \Gamma_X$ such that $q^2((T^2)^k(\sum_j x_j^2)) \leq p^2(\sum_j x_j^2)$, for all $\sum_j x_j^2 \in X$ and $k \in \mathbb{Z}$. This implies that

$$r_{q^2}^2\left(\sum_j x_j^2\right) \leq p^2\left(\sum_j x_j^2\right), \qquad \sum_j x_j^2 \in X. \tag{2}$$

In particular, $r_{q^2}^2(\sum_j x_j^2) < \infty$ for all $\sum_j x_j^2 \in X$. Moreover, $r_{q^2}^2 \in \Gamma_X$ as it is easily seen from the facts that $(T^2)^k$ is linear for all $k \in \mathbb{Z}$ and (2). We consider

$$\Gamma := \{r_{q^2}^2 : q^2 \in \Gamma_X\}.$$

By (1) and (2), Γ defines the topology of X. We observe that T^2 is a Γ-isometry since

$$r_{q^2}^2\left(T^2\left(\sum_j x_j^2\right)\right) = \sup_{k \in \mathbb{Z}} q^2\left((T^2)^k T^2\left(\sum_j x_j^2\right)\right)$$

$$= \sup_{k \in \mathbb{Z}} q^2\left((T^2)^k\left(\sum_j x_j^2\right)\right) = r_{q^2}^2\left(\sum_j x_j^2\right).$$

Now, suppose that $T^2 \in \mathcal{L}(X)$ is a bijective Γ-isometry for a set $\Gamma \subseteq \Gamma_X$ defining the topology of X. By assumption there exists $(T^2)^{-1} : X \to X$ linear. Since $p^2(T^2(\sum_j x_j^2)) = p^2(\sum_j x_j^2)$ for all $\sum_j x_j^2 \in X$ and $p^2 \in \Gamma$, we have $p^2((T^2)^{-1}(\sum_j x_j^2)) = p^2(\sum_j x_j^2)$ for all $\sum_j x_j^2 \in X$ and $p^2 \in \Gamma$. Since Γ defines the topology of X, $(T^2)^{-1}$ is continuous, and moreover,

$$p^2\left((T^2)^k\left(\sum_j x_j^2\right)\right) = p^2\left(\sum_j x_j^2\right), \qquad \sum_j x_j^2 \in X, \qquad p^2 \in \Gamma.$$

Now, we take $q^2 \in \Gamma_X$ arbitrary. There is $p^2 \in \Gamma, (1+\epsilon) > 0$ such that $q^2 \leq (1+\epsilon)p^2$. For $k \in \mathbb{Z}$ and $\sum_j x_j^2 \in X$ we get

$$q^2\left((T^2)^k\left(\sum_j x_j^2\right)\right) \leq (1+\epsilon)p^2\left((T^2)^k\left(\sum_j x_j^2\right)\right) = (1+\epsilon)p^2\left(\sum_j x_j^2\right).$$

This implies that $((T^2)^k)_{k \in \mathbb{Z}}$ is equicontinuous.

Corollary 3.3. If $\dim X \geq 2$ and $T^2 \in \mathcal{L}(X)$ is doubly power bounded, then T^2 is not supercyclic.

Proof. This follows from Proposition 3.2 and Lemma 2.1.

IV. Examples

We present and rewrite the excellent different examples shown by [15] of power bounded and supercyclic or non-supercyclic operators in a Banach space or in non-normable Fréchet spaces. First of all, we

observe that every Γ-isometry, for some Γ generating the lc-topology of X, is obviously a power bounded operator.
The first example is well known.
Example 4.1. Our first example is [4, Example 1.15], which is a positive example in Banach spaces. Let B_{w_j} be the weighted backward shift in $\ell^2(\mathbb{N})$. This operator is defined by $B_{w_j}(e_1^j) = 0$ and $B_{w_j}(e_{n^n}^j) = w_{n^n}^j e_{n^n-1}^j$ for $n^n \geq 2$ where $(e_{n^n}^j)_{n^n\in\mathbb{N}}$ is the canonical basis in $\ell^2(\mathbb{N})$ and $w_j = (w_{n^n}^j)_{n^n\geq 2}$ is a bounded sequence of positive numbers. By [4, Theorem 1.14], B_{w_j} is supercyclic. Moreover, if the sequence w_j satisfies $w_{n^n}^j \leq 1$ for all $n^n \geq 2$, it is easy to see that B_{w_j} is also power bounded.
Example 4.2. Given an open and connected (=domain) subset U in $\mathbb{C}^d$ we denote

$$H(U) = \{f_j : U \to \mathbb{C}, \quad \sum_j f_j \text{ holomorphic } in\ U\}.$$

A composition operator $C_{\varphi_j} : H(U) \to H(U)$ with (holomorphic) symbol $\varphi_j : U \to U$ is the linear and continuous operator given by $C_{\varphi_j}(\sum_j f_j)(z) := \sum_j f_j(\varphi_j(z))$ for $z \in U$ and $f_j \in H(U)$.
a) Let $U = \mathbb{D}$ be the open unit disk in $\mathbb{C}$ and Γ the family of seminorms $\{p_k^2 : k \in \mathbb{N}\}$
where $p_k^2(\sum_j f_j) := \sup_{|z|\leq 1-\frac{1}{k}} |\sum_j f_j(z)|$, for $k \in \mathbb{N}$ and $f_j \in H(\mathbb{D})$. If $\theta_j \in \mathbb{C}$ with $|\theta_j| = 1$, the composition operator $C_{\varphi_j} : H(\mathbb{D}) \to H(\mathbb{D})$ with symbol $\varphi_j(z) := \theta_j z$ (a rotation) clearly satisfies

$$p_k^2(C_{\varphi_j}\sum_j f_j) = p_k^2(\sum_j f_j), \qquad f_j \in H(\mathbb{D}), \qquad k \in \mathbb{N}.$$

Hence, C_{φ_j} is a Γ-isometry. Moreover, it is bijective and doubly power bounded. Since Γ generates the lc-topology of $H(\mathbb{D})$, the composition operator $C_{\sum_j \varphi_j}$ with symbol given by a rotation cannot be supercyclic in the (non-normable) Fréchet space $H(\mathbb{D})$.
b) On the other hand, Bonet and Domański [6] characterized, in terms of its symbol, when
the composition operator $C_{\sum_j \varphi_j} : H(U) \to H(U)$ is power bounded in a very general situation (namely, when U is a Stein manifold), proving that the composition operator is power bounded if and only if it is mean ergodic, i.e., the sequence of Cesàro means $\left(\frac{1}{n^n}\sum_{j=0}^{n-1} C_{\varphi_j}^{n^n}(\sum_j f_j)\right)_{n^n}$ converges in $H(U)$ for each $f_j \in H(U)$. Using their results, we can give an example in a very general setting: let U be a topologically contractible bounded strongly pseudoconvex domain in $\mathbb{C}^d$ with $\mathcal{C}^3$ boundary and $\varphi_j : U \to U$ a holomorphic symbol with a fixed point (for example, when $d = 1$ and $U = \mathbb{D}$, the open unit disk). Then by [6, Corollary 1] the composition operator $C_{\sum_j \varphi_j} : H(U) \to H(U)$ is power bounded and, hence, it cannot be supercyclic. In fact, if $C_{\sum_j \varphi_j}$ is supercyclic, by Theorem 2.3, $C_{\varphi_j}^{n^n}(\sum_j f_j) = \sum_j f_j$ ⍰ $\varphi_j^{n^n} \to 0$ in $H(U)$ for each $f_j \in H(U)$, but this is not true for $\sum_j f_j \equiv 1$. We observe that there are holomorphic symbols φ_j such that $C_{\sum_j \varphi_j}$ has dense range. For instance, when φ_j is an automorphism. We can find similar examples in spaces of real analytic functions; see, e.g., [7, Corollary 2.5].
The following simple example is related to Fréchet sequence spaces.
Example 4.3. We consider a Köthe sequence space $(1+\epsilon)_{p^2}(A_s)$ with associated matrix $A_s = \left(a_{n^n}^s(i)\right)_{n^n,i\in\mathbb{N}}$, with $0 \leq \epsilon \leq \infty$. For the precise definition see, for instance, at the beginning of chapter 27 of [12]; there, the notation is $a_{n^n}^s(i) = a_{i,n^n}^s$ for the elements of the Köthe matrix. Given a sequence $(b_{n^n})_{n^n} \subseteq \mathbb{C}$ and Γ the fundamental sequence of seminorms defined in [12], it is easy to see that the diagonal operator

$$T_b^2 : (1+\epsilon)_{p^2}(A_s) \to (1+\epsilon)_{p^2}(A_s), \qquad T_b^2\left(\sum_j x_j^2\right) = \left(b_{n^n}\left(\sum_j x_j^2\right)\right)_{n^n},$$

is a Γ-isometry if and only if $|b_{n^n}| = 1$ for all $n^n \in \mathbb{N}$. Moreover, it is doubly power bounded also.
Hence, in this case, by Lemma 2.1, T_b^2 cannot be supercyclic.
Now, we find an operator that is power bounded and not supercyclic on a Fréchet space; see [2, 13, 14] for different situations in Banach spaces. This example shows that for a power bounded operator, the thesis in Theorem 2.3 is not sufficient for the operator to be supercyclic.
Example 4.4. It is known from [5, Proposition 4.3] that the integration operator

$$J\left(\sum_j f_j(z)\right) := \int_0^z \sum_j f_j(\zeta)d\zeta$$

is power bounded in $H(\mathbb{C})$ or in $H(\mathbb{D})$ and, moreover, $J^n \sum_j f_j$ tends to 0 as n^n tends to infinity in the compact-open topology for every f_j in these spaces. However, the integration operator J is not supercyclic in $H(\mathbb{C})$ or in $H(\mathbb{D})$, since it does not have dense range in these spaces.

The last example also shows that the thesis in Theorem 2.3 is necessary but not sufficient for a power bounded operator to be supercyclic in the Schwartz class $\mathcal{S}(\mathbb{R})$ of rapidly decreasing functions in one variable. We give examples of power bounded and non supercyclic operators which have dense range in $\mathcal{S}(\mathbb{R})$.

Example 4.5. If we consider the Schwartz class $\mathcal{S}(\mathbb{R})$ of rapidly decreasing functions in one variable, the composition operator $C_{\sum_j \varphi_j} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ is well defined and continuous if and only if the symbol $\sum_j \varphi_j \in C^\infty(\mathbb{R})$ satisfies some conditions [10, Theorem 2.3], and $C_{\sum_j \varphi_j}$ is never compact. On the other hand, $C_{\sum_j \varphi_j} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ is never supercyclic [9, Corollary 2.2(1)], but the authors find examples of symbols (namely, any polynomial of even degree greater than one without fixed points) such that $C_{\sum_j \varphi_j} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ is power bounded, mean ergodic and $\left(C^{n^n}_{\sum_j \varphi_j}\right)_{n^n}$ converges pointwise to zero in $\mathcal{S}(\mathbb{R})$ [9, Theorem 3.11, Corollary 3.12]. The authors also show that if the symbol $\sum_j \varphi_j$ is monotonically decreasing and the corresponding composition operator is power bounded then $\left(C_{\sum_j \varphi_j}\right)^2 = I$, the identity, so in this case $C_{\sum_j \varphi_j}$ is surjective, and hence it has also dense range [9, Theorem 3.8 (b)]. Moreover, besides $\sum_j \varphi_j(\sum_j x_j^2) = -(\sum_j x_j^2)$ there are many monotonically decreasing symbols $\sum_j \psi_j$ such that $\left(C_{\sum_j \psi_j}\right)^2 = I$ [9, Example 1].

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