# Using Partitioned Design Matrices in Analyzing Nested-Factorial Experiments

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**Abstract.** Using QR Decomposition to calculate the sum of squares of a model has a limitation that the number of rows, which is also the number of observations or responses, has to be greater than the total number of parameters used in the model. The main goal in the experimental design model, as a part of the Linear Model, is to analyze the estimable function of the parameters used in the model. In order not to deal with generalized invers, partitioned design matrix may be used instead. This partitioned design matrix method may be used to calculate the sum of squares of the models whenever the total number of parameters is greater than the number of observations. It can also be used to find the degrees of freedom of each source of variation components. This method is discussed in a Balanced Nested-Factorial Experimental Design.

***Key words*** *: partitioned design matrices, nested-factorial, sum of squares, degrees of freedom*

## I. INTRODUCTION

A factorial experiment is an experiment in which all levels of a given factor are combined with all levels of every other factor in the experiment. There will be an interaction between the two factors when a change in one factor produces a different change in the response variabel at one level of another factor than at other levels of this factor. Some of the advantages of a factorial experiment can be seen : (1) more efficiency is possible than with one-factor-at-a-time experiments, (2) all data are used in computing both effects, (3) some information is gleaned on possible interaction between the two factors (Hicks, 1982)

In a factorial experiment the effects of a number of different factors are investigated simultaneously. The treatments consist of all combinations that can be formed from different factors. Some instances where factorial experimentation may be suitable : (1) in exploratory work where the object is to determine quickly the effects of each of a number of factors over a specified range, (2) in investigations of the interactions among the effects of several factors, since information is best obtained by testing all combinations, (3) in experiments designed to lead to recommendations that must apply over a wide range of conditions (Cochran & Cox, 1957).

Factorial treatment structure exists when the g treatments are the combinations of the levels of two or more factors. We call these combination treatments factor-level combinations or factorial combinations to emphasize that each treatment is a combination of one level of each of the factors. (Oehlert, 2010).

A factor is said to be nested within a second factor if each of its levels is observed in conjunction with just one level of the second factor. Nested factors are usually, but not always, random effects, and they are usually, but not always, blocking factors (Dean & Voss, 1999). If each level of factor A contains different levels of factor B, we say that factor B is nested within factor A. Experiments with two or more factors satisfying this definition are called nested factor experiments or nested design factor (Lentner & Bishop, 1986).

The factorial and nested designs are mostly used in manufacturing, processing, and the improvement of products (Longford, 1987; Rundan and Searle, 1991; Townsend and Searle, 1971)

QR Decomposition could be used to find the sum of squares of its source of variation's components as long as the number of rows of the matrix is at least equal to the number of columns. Thus, in the balanced nested-factorial design QR, decomposition cannot be used whenever $rabc < 1+a+b+ab+bc+abc$. Partitioning the design matrix with respect to source of variation's component could help calculating their sum of squares and determining the rank of its partitioned matrices which are also their degrees of freedom. Using classical-sigma notation to calculate sum of squares of each source of variation is not as easy as using the partitioned matrix method to do so.

## II. NOTATION

Consider the Balanced Nested-Factorial model as follows

$$Y_{ijkl} = \mu + \alpha_i + \beta_j + \alpha\beta_{ij} + \gamma(\beta)_{k(j)} + \alpha\gamma(\beta)_{ik(j)} + \varepsilon_{l(ijk)} \tag{1}$$

where $Y_{ijkl}$ = the *l-th* observation having *i-th* level effect of factor A, *j-th* level effect of facor B, and *k-th* level effect of factor C which is nested in *j-th* level of factor B; $\mu$ is the overall mean; $\alpha_i$ the effect of *i-th* level of factor A $i=1,2,\ldots,a$; $\beta_j$ the effect of *j*-th level of factor B $j=1,2,\ldots,b$; $\alpha\beta_{ij}$ the effect of interaction *i*-th level of factor A and *j*-th level of factor B; $\gamma(\beta)_{k(j)}$ the effect of *k*-th level of factor C which is nested in the *j*-th level of factor B; $k=1,2,\ldots,c$, $\alpha\gamma(\beta)_{ik(j)}$ the interaction component for *i*-th level of A and *k*-th level of C nested in *j*-th level of B; and $\varepsilon_{l(ijk)}$ the experimental error $l=1,2,\ldots,r$. It is assumed that $\varepsilon_{l(ijk)}$'s are i.i.d. $N(0,\sigma_\varepsilon^2)$, $\gamma(\beta)_{k(j)}$'s are i.i.d. $N(0,\sigma_\gamma^2)$, and $\alpha\gamma(\beta)_{ik(j)}$'s are i.i.d. $N(0,\sigma_{\alpha\gamma}^2)$. $\varepsilon_{l(ijk)}$'s, $\gamma(\beta)_{k(j)}$'s and $\alpha\gamma(\beta)_{ik(j)}$'s are mutually independents.

The sum of squares calculations for a Balanced Nested-Factorial are

$$CF = \frac{\left(\sum_{i,j,k,l} Y_{ijkl}\right)^2}{rabc} = \text{correction factor}$$

$$TotalSS = \sum_{i,j,k,l} Y_{ijkl}^2 - CF$$

$$SS[A] = \frac{1}{rbc}\sum_i Y_{i\ldots}^2 - CF$$

$$SS[B] = \frac{1}{rac}\sum_j Y_{.j..}^2 - CF \tag{2}$$

$$SS[AB] = \frac{1}{rc}\sum_{i.j} Y_{ij..}^2 - CF - SS\ for\ A - SS\ for\ B$$

$$SS[C(B)] = \frac{1}{rab}\sum_k Y_{..k.}^2 - CF - SS\ for\ B$$

$$SS[AC(B)] = \sum_j\left(\frac{1}{r}\sum_{i,k} Y_{ijk.}^2 - \frac{1}{rc}\sum_i Y_{ij..}^2 - \frac{1}{ra}\sum_k Y_{.jk.}^2 + \frac{Y_{.j..}^2}{rac}\right)$$

$$ErrorSS = TotalSS - SS[A] - SS[B] - SS[AB] - SS[C(B)\} - SS[AC(B)]$$

## III. LINEAR MODELS IN MATRIX NOTATION

A typical linear model considered is

$$\underline{Y} = X\underline{\beta} + \underline{\varepsilon} \tag{3}$$

where $\underline{Y}$ is an *n x 1* vector of observations, *X* is an *n x p* matrix of known constants called the design matrix, $\underline{\beta}$ is a *p x 1* vector of unobservable parameters, and $\underline{\varepsilon}$ is an *n x 1* vector of unobservable random errors. It is assumed that $E(\underline{\varepsilon}) = \underline{0}$ and Cov($\underline{\varepsilon}$) = $\sigma^2 I$ .

The design matrix *X* for the balanced nested-factorial experiments in (3) having size *rabc x (1+a+b+ab+bc+abc)*, is partitioned according to its source of variation components : Constants, Main Effect of Factor A, Main Effect of Factor B, Interaction Effect of Factor A and B, Main Effect of Factor C which is nested in Factor B, and Interaction Effect of Factor A and C which is nested in Factor B. Let $X_\mu = \underline{1}_{a\times1}\otimes\underline{1}_{b\times1}\otimes\underline{1}_{c\times1}\otimes\underline{1}_{r\times1}$ be the constant design matrix, $X_\alpha = I_{a\times a}\otimes\underline{1}_{b\times1}\otimes\underline{1}_{c\times1}\otimes\underline{1}_{r\times1}$ the main effect of factor A design matrix, $X_\beta = \underline{1}_{a\times1}\otimes I_{b\times b}\otimes\underline{1}_{c\times1}\otimes\underline{1}_{r\times1}$ the main effect of factor B design matrix, $X_{\alpha\beta} = I_{a\times a}\otimes I_{b\times b}\otimes\underline{1}_{c\times1}\otimes\underline{1}_{r\times1}$ the interaction effect between factor A and factor B design matrix, $X_\gamma = \underline{1}_{a\times1}\otimes I_{b\times b}\otimes I_{c\times c}\otimes\underline{1}_{r\times1}$ the main effect of factor C which is nested in factor B design matrix, and

$X_{\alpha\gamma} = I_{a\times a} \otimes I_{b\times b} \otimes I_{c\times c} \otimes \underline{1}_{r\times 1}$ the interaction effect of factor A and factor C which is nested in factor B design matrix, then $X = \left[X_\mu \mid X_\alpha \mid X_\beta \mid X_{\alpha\beta} \mid X_\gamma \mid X_{\alpha\gamma}\right]$.

Furthermore, the projection matrix has the form of $M_* = X_*(X_*^t X_*)^{-1} X_*^t$. Therefore, for each partitioned design matrix, it is easily to verify that the projection matrices with respect to each of the source of variations are as follows:

$M_\mu = \frac{1}{rabc} J_{a\times a} \otimes J_{b\times b} \otimes J_{c\times c} \otimes J_{r\times r}$,

$M_\alpha = \frac{1}{rbc} I_{a\times a} \otimes J_{b\times b} \otimes J_{c\times c} \otimes J_{r\times r}$,

$M_\beta = \frac{1}{rac} J_{a\times a} \otimes I_{b\times b} \otimes J_{c\times c} \otimes J_{r\times r}$,

$M_{\alpha\beta} = \frac{1}{rc} I_{a\times a} \otimes I_{b\times b} \otimes J_{c\times c} \otimes J_{r\times r}$,

$M_\gamma = \frac{1}{ra} J_{a\times a} \otimes I_{b\times b} \otimes I_{c\times c} \otimes J_{r\times r}$, and

$M_{\alpha\gamma} = \frac{1}{r} I_{a\times a} \otimes I_{b\times b} \otimes I_{c\times c} \otimes J_{r\times r}$

Using the properties of Kronecker product, we can easily find simple form every combination of projection matrix multiplication. The results of these multiplications are summarized as in Table 1.

## IV. SUM OF SQUARES IN A MATRIX NOTATION

In terms of matrix notation, the formulas for calculating sum of squares for balanced nested-factorial experiment as mentioned in (2), can be written as follows:

*SS[A]* = $\underline{Y}'(M_\alpha - M_\mu)\underline{Y}$,

*SS[B]* = $\underline{Y}'(M_\beta - M_\mu)\underline{Y}$,

*SS[AB]* = $\underline{Y}'(M_{\alpha\beta} - M_\alpha - M_\beta + M_\mu)\underline{Y}$,

*SS[C(B)]* = $\underline{Y}'(M_\gamma - M_\beta)\underline{Y}$,

*SS[AC(B)]* = $\underline{Y}'(M_{\alpha\gamma} - M_{\alpha\beta} - M_\gamma + M_\beta)\underline{Y}$,

*Error SS* = $\underline{Y}'(I - M_{\alpha\gamma})\underline{Y}$ and

*Total SS* = $\underline{Y}'(I - M_\mu)\underline{Y}$.

Let the *n x 1* random vector $\underline{Y}$ be distributed N($\underline{y}$ : $\underline{\mu}$, I). The random variable $U = Y'AY$ is distributed as $\chi^2$ (u:*K*;*λ*) with *K* degrees of freedom and noncentrality parameter *λ*, where $\lambda = \mu' A\mu/2$, if and only if *A* is an idempotent matrix of rank *K* (Graybill, 2000).

It can be easily verified, using the result presented in Table 1, that $M_\alpha - M_\mu$, $M_\beta - M_\mu$, $M_{\alpha\beta} - M_\alpha - M_\beta + M_\mu$, $M_\gamma - M_\beta$, $M_{\alpha\gamma} - M_{\alpha\beta} - M_\gamma + M_\beta$ and $I - M_{\alpha\gamma}$ are all idempotent matrices. In additions to those idempotent properties, they are also symmetric matrices. From the properties of symmetric and idempotent matrices, their ranks are just equal to their traces (Rencher and Schaalje, 2008). Thus, their ranks are

$tr(M_\alpha - M_\mu) = a\text{-}1$,

$tr(M_\beta - M_\mu) = b\text{-}1$,

$tr(M_{\alpha\beta} - M_\alpha - M_\beta + M_\mu) = ab\text{-}a\text{-}b\text{+}1 = (a\text{-}1)(b\text{-}1)$,

$tr(M_\gamma - M_\beta) = bc\text{-}b = b(c\text{-}1)$,

$tr(M_{\alpha\gamma} - M_{\alpha\beta} - M_\gamma + M_\beta) = abc\text{-}ab\text{-}bc\text{+}b = (a\text{-}1)b(c\text{-}1)$ and

$tr(I - M_{\alpha\gamma}) = rabc\text{-}abc = (r\text{-}1)abc$ respectively.

Using the above arguments, and without loss of generality that the random vector $\underline{Y}$ be distributed N($\underline{y}$ : $\underline{0}$, I), therefore the distributions of sum of squares are as follows:

*SS[A]* is distributed as $\chi^2_{a-1}$, *SS[B]* is distributed as $\chi^2_{b-1}$,

*SS[AB]* is distributed as $\chi^2_{(a-1)(b-1)}$,

*SS[C(B)]* is distributed as $\chi^2_{b(c-1)}$,

*SS[AC(B)]* is distributed as $\chi^2_{(a-1)b(c-1)}$, and

*Error SS* is distributed as $\chi^2_{(r-1)abc}$.

## V. HYPOTHESIS TESTING

Mean of Square is defined as Sum of Square divided by its degrees of freedom. Therefore, we have the followings :

*MS[A]* = $\underline{Y}'(M_\alpha - M_\mu)\underline{Y}/(a-1)$,

*MS[B]* = $\underline{Y}'(M_\beta - M_\mu)\underline{Y}/(b-1)$,

*MS[AB]* = $\underline{Y}'(M_{\alpha\beta} - M_\alpha - M_\beta + M_\mu)\underline{Y}/((a-1)(b-1))$,

*MS[C(B)]* = $\underline{Y}'(M_\gamma - M_\beta)\underline{Y}/(b(c-1))$,

*MS[AC(B)]* = $\underline{Y}'(M_{\alpha\gamma} - M_{\alpha\beta} - M_\gamma + M_\beta)\underline{Y}/((a-1)b(c-1))$ and

*MS[Error]* = $\underline{Y}'(I - M_{\alpha\gamma})\underline{Y}/((r-1)abc)$.

We need to know first the expected mean squares (EMS) for each of the source of variation in the balanced nested-factorial design.

*EMS for Error* = $\sigma^2_\varepsilon$,

*EMS for AC(B)* = $\sigma^2_\varepsilon + r\sigma^2_{\alpha\gamma}$,

*EMS for C(B)* = $\sigma^2_\varepsilon + ra\sigma^2_\gamma$,

*EMS for AB* = $\sigma^2_\varepsilon + r\sigma^2_{\alpha\gamma} + \underline{Y}'(M_{\alpha\beta} - M_\alpha - M_\beta + M_\mu)\underline{Y}/((a-1)(b-1))$,

*EMS for B* = $\sigma^2_\varepsilon + ra\sigma^2_\gamma + \underline{Y}'(M_\beta - M_\mu)\underline{Y}/(b-1)$ dan

*EMS for A* = $\sigma^2_\varepsilon + r\sigma^2_{\alpha\gamma} + \underline{Y}'(M_\alpha - M_\mu)\underline{Y}/(a-1)$.

We know that if W is distributed as chi-square with *a* degrees of freedom, Z is distributed as chi-square with *b* degrees of freedom, W and Z are independent to each other, then (W/*a*)/(Z/*b*) is distributed as F with *a* and *b* degrees of freedom.

To check the independence of two matrices W and Z, we need to show that WZ = O. Using the information in Table 1, it is easy to verify that $(M_{\alpha\gamma} - M_{\alpha\beta} - M_\gamma + M_\beta)(I - M_{\alpha\gamma}) = 0$, therefore $(M_{\alpha\gamma} - M_{\alpha\beta} - M_\gamma + M_\beta)$ and $(I - M_{\alpha\gamma})$ are independent; $(M_\gamma - M_\beta)(I - M_{\alpha\gamma}) = 0$, therefore $(M_\gamma - M_\beta)$ and $(I - M_{\alpha\gamma})$ are independent; $(M_{\alpha\beta} - M_\alpha - M_\beta + M_\mu)(M_{\alpha\gamma} - M_{\alpha\beta} - M_\gamma + M_\beta) = 0$, therefore $(M_{\alpha\beta} - M_\alpha - M_\beta + M_\mu)$ and $(M_{\alpha\gamma} - M_{\alpha\beta} - M_\gamma + M_\beta)$ are independent; $(M_\beta - M_\mu)(M_\gamma - M_\beta) = 0$, therefore $(M_\beta - M_\mu)$ and $(M_\gamma - M_\beta)$ are independent; and $(M_\alpha - M_\mu)(M_{\alpha\gamma} - M_{\alpha\beta} - M_\gamma + M_\beta) = 0$, therefore $(M_\alpha - M_\mu)$ and $(M_{\alpha\gamma} - M_{\alpha\beta} - M_\gamma + M_\beta)$ are independent also.

Therefore, we have the following results :

- to test if there is a significant main effect of factor A is to reject the null hypothesis whenever *MS[A]/MS[AC(B)]* is large enough. *MS[A]/MS[AC(B)]* is distributed as F with *a*-1 and *(a-1)b(c-1)* degrees of freedom.
- to test if there is a significant main effect of factor B is to reject the null hypothesis whenever *MS[B]/MS[C(B)]* is large enough. *MS[B]/MS[C(B)]* is distributed as F with *b*-1 and *b(c-1)* degrees of freedom.
- to test if there is a significant AB interaction effect is to reject the null hypothesis whenever *MS[AB]/MS[AC(B)]* is large enough. *MS[AB]/MS[AC(B)]* is distributed as F with *(a-1)(b-1)* and *(a-1)b(c-1)* degrees of freedom.

- to test if there is a significant variance of C which is nested in B is to reject the null hypothesis whenever *MS[C(B)]/MS[Error]* is large enough. *MS[C(B)]/MS[Error]* is distributed as F with *b(c-1)* and *(r-1)abc* degrees of freedom.
- to test if there is a significant variance of AC interaction where C is nested in B is to reject the null hypothesis whenever *MS[AC(B)]/MS[Error]* is large enough. *MS[AC(B)]/MS[Error]* is distributed as F with *(a*-1)*b(c-1)* and *(r-1)abc* degrees of freedom.

## VI. EXAMPLE

R routine which can be used to analyze balanced nested-factorial experimental data is provided in the Appendix. The following example is taken from Hicks (1982) : An investigator wished to improve the number of rounds per minute that could be fired from a Navy gun. He devised a new loading method, which he hoped would increase the number of rounds per minute when compared with the existing method of loading. To test this hypothesis he needed teams of men to operate the equipment. As the general physique of a man might affect the speed with which he could handle the loading of the gun, he chose teams of men in three general groupings- slight, average, and heavy or rugged men. The classification of such men was on the basis of an Armed Services classification table. He chose three teams at random to represent each of three physique groupings. Each team was presented with the two methods of gun loading in a random order and each team used each method twice.

| Group | | I | | | II | | | III | |
|---|---|---|---|---|---|---|---|---|---|
| Team | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| Method 1 | 20.2 | 26.2 | 23.8 | 22.0 | 22.6 | 22.9 | 23.1 | 22.9 | 21.8 |
| | 24.1 | 26.9 | 24.9 | 23.5 | 24.6 | 25.0 | 22.9 | 23.7 | 23.5 |
| Method 2 | 14.2 | 18.0 | 12.5 | 14.1 | 14.0 | 13.7 | 14.1 | 12.2 | 12.7 |
| | 16.2 | 19.1 | 15.4 | 16.1 | 18.1 | 16.0 | 16.1 | 13.8 | 15.1 |

The R routine results the following output.

| | [,1] | [,2] | [,3] | [ ,4] | [,5] |
|---|---|---|---|---|---|
| [1,] | "Source" | "Deg Frdm" | "SS" | "MS" | "F" |
| [2,] | "Method" | "1" | 651.951" | "651.951" | "364.83" |
| [3,] | "Group" | "2" | "16.052" | "8.026" | "1.227" |
| [4,] | "Method x Group" | "2" | "1.187" | "0.594" | "0.332" |
| [5,] | "Team(Group)" | "6" | "39.258" | "6.543" | "2.831" |
| [6,] | "Method x Team(Group)" | "6" | "10.722" | "1.787" | "0.773" |
| [7,] | "Error" | "18" | "41.59" | "2.311" | "" |
| [8,] | "Total" | "35" | "760.76" | "" | "" |

Note that
$F_{0.05;1;6} = 5.98$, $F_{0.05;2;6} = 5.14$, $F_{0.05;6;18} = 2.66$
$F_{0.01;1;6} = 13.74$, $F_{0.01;2;6} = 10.92$, $F_{0.01;6;18} = 4.01$

There are significant differences in Method effects at 1% level, and significant variance of Team effects which is nested in Group at 5% level, while the other source of variations are not significantly different.

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## APPENDIX

```
##### R routine for Balanced Nested-Factorial Experiments #####

##### Given the Data #####
#### a is the level size of factor A (Methods)
#### b is the level size of factor B (Groups)
#### c is the level size of factor C (Teams) which is nested in factor B (Groups)
#### r is the number of replications
a <-2
b <-3
c <-3
r <-2

##### Observed Data #####
y <-rbind(20.2, 24.1, 26.2, 26.9, 23.8, 24.9, 22.0, 23.5, 22.6,
        24.6, 22.9, 25.0, 23.1, 22.9, 22.9, 23.7, 21.8, 23.5,
        14.2, 16.2, 18.0, 19.1, 12.5, 15.4, 14.1, 16.1, 14.0,
        18.1, 13.7, 16.0, 14.1, 16.1, 12.2, 13.8, 12.7, 15.1)

##### Basic Vectors and Matrices #####
va <- matrix(1,a,1) #vektor 1a
vb <- matrix(1,b,1) #vektor 1b
vc <- matrix(1,c,1) #vektor 1c
vr <- matrix(1,r,1) #vektor 1r
Ia <- diag(1,a,a) #identitas a
Ib <- diag(1,b,b) #identitas b
Ic <- diag(1,c,c) #identitas c

##### Partitioned Design Matrices #####
Xmu <- kronecker(va,kronecker(vb,kronecker(vc,vr)))
Xm <- kronecker(Ia,kronecker(vb,kronecker(vc,vr)))
Xg <- kronecker(va,kronecker(Ib,kronecker(vc,vr)))
Xmg <- kronecker(Ia, kronecker(Ib,kronecker(vc,vr)))
Xt <- kronecker(va,kronecker(Ib,kronecker(Ic,vr)))
Xmt <- kronecker(Ia,kronecker(Ib,kronecker(Ic,vr)))

##### Projection Matrices #####
Mmu <-(Xmu %*%(solve(t(Xmu)%*%Xmu)))%*% t(Xmu)
Mm <-(Xm %*%(solve(t(Xm)%*%Xm)))%*% t(Xm)
Mg <-(Xg %*%(solve(t(Xg)%*%Xg)))%*% t(Xg)
Mmg <-(Xmg %*%(solve(t(Xmg)%*%Xmg)))%*% t(Xmg)
Mt <-(Xt %*%(solve(t(Xt)%*%Xt)))%*% t(Xt)
Mmt <-(Xmt %*%(solve(t(Xmt)%*%Xmt)))%*% t(Xmt)
```

```
##### Calculating Sum of Squares #####
SSMethod <- round(t(y)%*%(Mm-Mmu)%*%y, digits=3)
SSGroup <- round(t(y)%*%(Mg-Mmu)%*%y, digits=3)
SSMethodxGroup <- round(t(y)%*%(Mmg-Mm-Mg+Mmu)%*%y,digits=3)
SSTeamwGroup <- round(t(y)%*%(Mt-Mg)%*%y, digits=3)
SSMethodxTeamwGroup <- round(t(y)%*%(Mmt-Mmg-Mt+Mg)%*%y,digits=3)
SSError <- round(t(y)%*%(diag(1,a*b*c*r,a*b*c*r)-Mmt)%*%y,digits=3)
SSTotal <- round(t(y)%*%(diag(1,a*b*c*r,a*b*c*r)-Mmu)%*%y,digits=3)

##### Calculating Means of Squares #####
library (psych)
MSMethod <- round(SSMethod/tr(Mm-Mmu),digits=3)
MSGroup <- round(SSGroup/tr(Mg-Mmu),digits=3)
MSMethodxGroup <- round(SSMethodxGroup/tr(Mmg-Mm-Mg+Mmu),digits=3)
MSTeamwGroup <- round(SSTeamwGroup/tr(Mt-Mg),digits=3)
MSMethodxTeamwGroup <- round(SSMethodxTeamwGroup/tr(Mmt-Mmg-Mt+Mg),digits=3)
MSError <- round(SSError/tr(diag(1,a*b*c*r,a*b*c*r)-Mmt),digits=3)
MSTotal <- round(SSTotal/tr(diag(1,a*b*c*r,a*b*c*r)-Mmu),digits=3)

##### Calculating F #####
FMethod <- round(MSMethod/MSMethodxTeamwGroup,digits=3)
FGroup <- round(MSGroup/MSTeamwGroup,digits=3)
FMethodxGroup <- round(MSMethodxGroup/MSMethodxTeamwGroup,digits=3)
FTeamwGroup <- round(MSTeamwGroup/MSError,digits=3)
FMethodxTeamwGroup <- round(MSMethodxTeamwGroup/MSError,digits=3)

##### Summary #####
sources<-rbind("Method","Group","MethodxGroup","Team(Group)","MethodxTeam(Group)",
"Error","Total")
Values <- cbind("Source","Deg Frdm","SS","MS","F")
SS1<-
rbind(SSMethod,SSGroup,SSMethodxGroup,SSTeamwGroup,SSMethodxTeamwGroup,SSError,SSTotal)
MS1<-
rbind(MSMethod,MSGroup,MSMethodxGroup,MSTeamwGroup,MSMethodxTeamwGroup,MSError,"")
DB1 <-rbind(tr(Mm-Mmu),tr(Mg-Mmu),tr(Mmg-Mm-Mg+Mmu),tr(Mt-Mg),tr(Mmt-Mmg-Mt+Mg),
        tr(diag(1,a*b*c*r,a*b*c*r)-Mmt),tr(diag(1,a*b*c*r,a*b*c*r)-Mmu))
F1 <- rbind(FMethod,FGroup,FMethodxGroup,FTeamwGroup,FMethodxTeamwGroup,"","")
nestedfactrabc <-cbind(sources,DB1,SS1,MS1,F1)
outnestedfactorial <-rbind(Values,nestedfactrabc)
outnestedfactorial
```

**Table 1. Matrix Multiplication**

| | $M_\mu$ | $M_\alpha$ | $M_\beta$ | $M_{\alpha\beta}$ | $M_\gamma$ | $M_{\alpha\gamma}$ |
|---|---|---|---|---|---|---|
| $M_\mu$ | $M_\mu$ | $M_\mu$ | $M_\mu$ | $M_\mu$ | $M_\mu$ | $M_\mu$ |
| $M_\alpha$ | $M_\mu$ | $M_\alpha$ | $M_\mu$ | $M_\alpha$ | $M_\mu$ | $M_\alpha$ |
| $M_\beta$ | $M_\mu$ | $M_\mu$ | $M_\beta$ | $M_\beta$ | $M_\beta$ | $M_\beta$ |
| $M_{\alpha\beta}$ | $M_\mu$ | $M_\alpha$ | $M_\beta$ | $M_{\alpha\beta}$ | $M_\beta$ | $M_{\alpha\beta}$ |
| $M_\gamma$ | $M_\mu$ | $M_\mu$ | $M_\beta$ | $M_\beta$ | $M_\gamma$ | $M_\gamma$ |
| $M_{\alpha\gamma}$ | $M_\mu$ | $M_\alpha$ | $M_\beta$ | $M_{\alpha\beta}$ | $M_\gamma$ | $M_{\alpha\gamma}$ |