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A Remark on the Hilbert Hardy Inequalities

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Abstract: We shall apply the Minkowski'a inequality in the exponent to derive a new and sharper results on Hilbert and Hardy inequalities.

Key Words: Holder's Inequality, Minkowski's Inequality, Hilbert-Hardy Inequality

I. Introduction

In 1952, [1] studied the Hilbert-Hardy inequality for positive functions f, g with $p > 1; \frac{1}{p} + \frac{1}{q} = 1$ as

$$\int_0^\infty \int_0^\infty \frac{f(x) g(y)}{x+y} dxdy < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left(\int_0^\infty f^p(x) dx\right)^{\frac{1}{p}} \left(\int_0^\infty g^q(y) dy\right)^{\frac{1}{q}}$$

(1.1)

provided that the integrals on the right hand side are convergent. The best possible constant derived is $\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}$.

In 2005, [2] discussed the inequality

$$\int_0^\infty \int_0^\infty \frac{f(x) g(y)}{(x+y)^\lambda} dxdy < B(1 - pA_2, \lambda + pA_2 - 1) \left(\int_0^\infty x^{pqA_1 - 1} f^p(x) dx\right)^{\frac{1}{p}} \left(\int_0^\infty y^{pqA_2 - 1} g^q(y) dy\right)^{\frac{1}{q}}$$

(1.2)

where $B(1 - pA_2, \lambda + pA_2 - 1)$ is a beta function and is the best possible constant,

$\lambda > 0,\ A_1 \in \left(\frac{1-\lambda}{q}, \frac{1}{q}\right), A_2 \in \left(\frac{1-\lambda}{p}, \frac{1}{p}\right)$ and $pA_2 + qA_1 = 2 - \lambda$.

In 2011, [3] studied the inequality

$$\int_a^b\int_a^b \frac{f(x)g(y)}{(\mathrm{au}^2(x)+2bu(x)v(y)+\mathrm{cv}^2(y))^{\lambda}}dxdy$$

$$< L^*\left(\int_0^{\infty}\frac{u(x)^{pqA_1-1}}{u'(x)^{p-1}}f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^{\infty}\frac{v(y)^{pqA_2-1}}{v'(y)^{q-1}}g^q(y)dy\right)^{\frac{1}{q}}$$

(1.3)

where $L^* = a^{\frac{qA_1-1}{2}}c^{\frac{pA_2-1}{2}}B(1-pA_2,2\lambda+pA_2-1)F\left(\frac{1-pA_2}{2},\lambda-\frac{1-pA_2}{2},\lambda+\frac{1}{2};1-\frac{b^2}{ac}\right)$ is the best possible constant, $F(a,b;c;x)$ is the hypergeometric function $\lambda>0$ $A_1\in\left(\frac{1-\lambda}{q},\frac{1}{q}\right), A_2\in\left(\frac{1-\lambda}{p},\frac{1}{p}\right)$ and $pA_2+qA_1=2-\lambda$, $a,c>0,\ b^2<ac,\ u$ and v are differentiable nonnegative strictly increasing functions on $(a,b),(-\infty\le a<b\le\infty)$, and they satisfy the following conditions:

$$\lim_{t\to a^+}u(t)=\lim_{t\to a^+}v(t)=0 \text{ and } \lim_{t\to b^-}u(t)=\lim_{t\to b^-}v(t)=\infty$$

In 2013 , [4] discussed the inequality

$$\int_a^b\int_a^b\frac{f(x)g(y)}{(u(x)+v(y))^{\lambda}}dxdy < c\left(\int_a^b[u(x)]^{-\lambda-1-p\gamma}u'(x)F^p(x)dx\right)^{\frac{1}{p}}$$
$$\times\left(\int_a^b[v(y)]^{-\lambda-1+q\gamma}v'(x)G^q(y)dy\right)^{\frac{1}{q}}$$

(1.4)

provided the integrals on the right hand side are convergent; where

$c=\left(\frac{\lambda}{p}+\gamma\right)\left(\frac{\lambda}{q}-\gamma\right)B\left(\frac{\lambda}{p}+\gamma,\frac{\lambda}{q}-\gamma\right)$ is the best possible constant with the following conditions assumed:

$p>1;\frac{1}{p}+\frac{1}{q}=1, \lambda>0, \gamma\in\left(\frac{-\lambda}{p},\frac{\lambda}{q}\right), f,g>0,\ f,g\in L(a,b)$ and $F(x)=\int_a^x f(\tau)d\tau$, $G(x)=\int_a^x g(\tau)d\tau$.

The reverse form of (1.4) was also studied.

II. Results

In this paper, equation (1.4) is discussed using the following properties:

1. $\left|(u+v)^p\right|\le\left|(u+v)\right|^p\le(|u|+|v|)^p\le 2^p\left(|u|^p+|v|^p\right)$

2. $$\frac{1}{(u+v)^{\lambda}} = \int_0^\infty e^{-(u+v)^{\lambda} t} dt \ge \int_0^\infty e^{-\left((2u)^{\lambda}+(2v)^{\lambda}\right)t} dt = \frac{1}{2^{\lambda}\left(u^{\lambda}+v^{\lambda}\right)}$$ provided $\lambda > 0, u(x), v(x) > 0$.

Theorem 1 Let $p>1; \frac{1}{p}+\frac{1}{q}=1, \alpha=\frac{\lambda}{pq}, \lambda>0; f,g>0; f,g \in L(a,b); \Phi(x)=\int_a^x f(\tau)d\tau$, $\Omega(y)=\int_a^y g(\tau)d\tau$ and let $\lim_{t\to a^+} u(t)=\lim_{t\to a^+} v(t)=0$ and $\lim_{t\to b^-} u(t)=\lim_{t\to b^-} v(t)=\infty$. If $\int_a^b [u(x)]^{-\lambda-1} u'(x)\Phi^p(x)dx<\infty, \int_a^b [v(y)]^{-\lambda-1} v'(y)\Omega^q(y)dy<\infty$, then

$$\int_a^b\int_a^b \frac{f(x)g(y)}{2^{\lambda}\left(u^{\lambda}+v^{\lambda}\right)}dxdy \le c\left(\int_a^b [u(x)]^{-\lambda-1} u'(x)\Phi^p(x)dx\right)^{\frac{1}{p}}$$

$$\times\left(\int_a^b [v(y)]^{-\lambda-1} v'(x)\Omega^q(y)dy\right)^{\frac{1}{q}} \le \int_a^b\int_a^b \frac{f(x)g(y)}{\left(u(x)+v(y)\right)^{\lambda}}dxdy$$

(2.1)

where $c=\frac{\lambda^2}{2^{\lambda}}\Gamma\left(\frac{\lambda}{p}+\frac{1}{q}\right)^{1/q}\Gamma\left(\frac{\lambda}{q}+\frac{1}{p}\right)^{1/p} B\left(\frac{\lambda}{p},\frac{1}{q}\right)^{1/q} B\left(\frac{\lambda}{q},\frac{1}{p}\right)^{1/p}$ and $B(m,n)$ is a beta function.

Proof On applying the Holder's inequality;

$$\int_a^b\int_a^b \frac{f(x)g(y)}{2^{\lambda}\left(u^{\lambda}(x)+v^{\lambda}(y)\right)}dxdy = \frac{1}{2^{\lambda}}\int_a^b\int_a^b\left(\int_0^\infty e^{-\left[u^{\lambda}(x)+v^{\lambda}(y)\right]t}dt\right)f(x)g(y)dxdy$$

$$= \frac{1}{2^{\lambda}}\int_0^\infty\left(\int_a^b\left(e^{-\left[u^{\lambda}(x)\right]t}f(x)dx\right)\left(\int_a^b e^{-\left[v^{\lambda}(y)\right]t}dy\right)\right)dt$$

$$\le \frac{1}{2^{\lambda}}\left(\int_0^\infty\int_a^b\left(e^{-\left[u^{\lambda}(x)\right]t}f(x)dx\right)^p dt\right)^{1/p}\left(\int_0^\infty\int_a^b\left(e^{-\left[v^{\lambda}(y)\right]t}g(y)dy\right)^q dt\right)^{1/q} \quad (2.2)$$

Now, we have

$$\int_a^b\left(e^{-\left[u^{\lambda}(x)\right]t}f(x)dx\right)=\lambda^{1/q}t^{1/q-\alpha/\lambda}\Gamma(p\alpha+1)^{1/q}\int_a^b\left(u^{-q\alpha}u^{\lambda-1}e^{-\left[u^{\lambda}(x)\right]t}\Phi^q(x)dx\right)^{1/q}$$

(2.3)

On interchanging roles of p and q we get;

a).

$$\left(\int_a^b\left(e^{-\left[u^{\lambda}(x)\right]t}f(x)dx\right)\right)^p=\lambda t^{1-\alpha p/\lambda}\Gamma(q\alpha+1)^{\frac{p}{q}}\int_a^b\left(u^{-p\alpha}(x)u^{\lambda-1}(x)u'(x)e^{-\left[u^{\lambda}(x)\right]t}\Phi^p(x)dx\right)$$

similarly for the second bracket in (2.2),

b).

$$\left(\int_a^b \left(e^{-\left[v^\lambda(y)\right]t} g(y)\,dy\right)\right)^q = \lambda t^{1-\alpha q/\lambda}\Gamma(q\alpha+1)^{\frac{q}{p}}\int_a^b \left(v^{-q\alpha}(y)v^{\lambda-1}(y)v'(y)e^{-\left[v^\lambda(y)\right]t}\Omega^q(y)\,dy\right)$$

substituting (a) and (b) in equation (2.2), and let we get

$$\int_a^b\int_a^b \frac{f(x)g(y)}{2^\lambda\left(u^\lambda(x)+v^\lambda(y)\right)}dxdy \le \frac{\lambda}{2^\lambda}\Gamma\left(\frac{\lambda}{p}+1\right)^{\frac{1}{q}}\Gamma\left(\frac{\lambda}{q}+1\right)^{\frac{1}{p}}\int_a^b\left(u^{-\lambda/q+\lambda-1}(x)u'(x)\Phi^p(x)\left(\int_0^\infty t^{1/p}e^{-\left[u^\lambda(x)\right]t}dt\right)dx\right)^{1/p}$$

$$\times\int_a^b\left(v^{-\lambda/p+\lambda-1}(y)v'(y)\Omega^q(y)\left(\int_0^\infty t^{1/q}e^{-\left[u^\lambda(x)\right]t}dt\right)dx\right)^{1/q}$$

$$=\frac{\lambda^2}{2^\lambda}\Gamma\left(\frac{\lambda}{p}+\frac{1}{q}\right)^{1/q}\Gamma\left(\frac{\lambda}{q}+\frac{1}{p}\right)^{1/p}B\left(\frac{\lambda}{p},\frac{1}{q}\right)^{1/q}B\left(\frac{\lambda}{q},\frac{1}{p}\right)^{1/p}$$

$$\times\left(\int_a^b\left[u(x)\right]^{-\lambda-1}u'(x)\Phi^p(x)\,dx\right)^{\frac{1}{p}}\left(\int_a^b\left[v(y)\right]^{-\lambda-1}v'(x)\Omega^q(y)\,dy\right)^{\frac{1}{q}}$$

(2.4)

The verification that the constant c is the best possible is similar to that done by [4].

III. Example

If we let $u(x)=x,\ v(y)=y,\ \lambda=1$, the equation (2.4) reduces to

$$\int_0^\infty\int_0^\infty\frac{f(x)g(y)}{(x+y)}dxdy \le B\left(\frac{1}{p},\frac{1}{q}\right)\left(\int_a^b\frac{\Phi^p(x)}{x^2}dx\right)^{\frac{1}{p}}\left(\int_a^b\frac{\Omega^q(y)}{y^2}dy\right)^{\frac{1}{q}}$$

$$=\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\left(\int_a^b\frac{\Phi^p(x)}{x^2}dx\right)^{\frac{1}{p}}\left(\int_a^b\frac{\Omega^q(y)}{y^2}dy\right)^{\frac{1}{q}}.$$

which coincides with [1].

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