*International Journal of Scientific and Technical Research in Engineering (IJSTRE)*
*www.ijstre.com Volume 8 Issue 6 ǁ Nov-Dec 2023.*
*ISSN: 2581-9941.*

# Odd gracefulness of some graphs in context of graph operations

[1]Divya K. Jadeja, [2]Krishna N. Gotecha and [3]Dr. V. J. Kaneria
[1,2,3]*Saurashtra University, Rajkot - 360005, India.*
[1] divyajadeja89@gmail.com, [2] krishna.gotecha99@gmail.com , [3] kaneriavinodray@gmail.com

***Abstract:*** *In this paper, we prove that every $\alpha$ - graceful graph is also odd graceful. Also, using graph operation union of different graph families such as $P_n \times P_m \cup P_r$ , $K_{m,n} \cup P_r$ ,$P_m \cup P_n$ and $K_{m,n} \cup P_r \cup P_t$ we prove that these are all odd graceful graphs for every $m, n, r, t \in \mathbb{N} - \{1\}$.*

***Keywords:*** *$\alpha$ -graceful graph, odd graceful graph, union of graphs, path, grid graph, complete bipartite graph.*

## I. Introduction

In this paper we begin with simple, finite and undirected graph $G = (V(G), E(G))$ with $p$ vertices and $q$ edges. For standard terminology and notations, we follow Harary. We will provide brief summary of definitions and other information which are prerequisites for the present investigations.

The study of graceful graphs and graceful labeling methods was introduced by Rosa. While the graceful labeling of graphs was perceived to be primarily theoretical subject in the field of graph theory and discrete mathematics. But it is now accepted that gracefully labeled graphs often serve as models in a wide range of applications. Such applications including coding theory and communication network addressing. The brief survey on applications of gracefully labeled graphs is reported in Bloom and Golomb. The famous graceful tree conjecture and many illustrious works brought a tide of labeling scheme having graceful theme. For detailed survey on graph labeling we refer to Gallian.

**Definition 1:** A graph labeling is an assignment of integers to the vertices or edges or both, subject to certain conditions which have been motivated by practical problems.

**Definition 2:** A function $f$ is called graceful labeling of graph $G$ if $f : V \to \{0, 1, \ldots, q\}$ is injective and the induced function $f : E \to \{1, 2, \ldots, q\}$ defined as $f(e = uv) = |f(u) - f(v)|$ is bijective. A graph which admits graceful labeling is called a graceful graph.

**Definition 3:** $\alpha$- labeling is a graceful labeling with an additional property that $\exists$ a non-negative integer $k (0 \leq k < q) \ni \min\{f(u), f(v)\} \leq k < \max\{f(u), f(v)\}, \forall uv \in E(G)$. A graph which admits a $\alpha$- labeling must be a bipartite graph. Graceful graph with $\alpha$- labeling is called $\alpha$- graceful graph.

**Definition 4:** A graph G = $(V(G), E(G))$ with $p$ vertices and $q$ edges is said to admit an odd graceful labeling if $f : V(G) \to \{0, 1, 2, \ldots, 2q - 1\}$ is injective and the induced function $f^* : E(G) \to \{1, 3, 5, \ldots, 2q - 1\}$ defined as $f^*(e = uv) = |f(u) - f(v)|$ is bijective. The graph which admits odd graceful labeling is called an odd graceful graph.

**Definition 5:** Given two graphs $G_1$ and $G_2$, their union will be a graph such that $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1) \cup E(G_2)$.

The above concept was introduced by Gnanajothi and in the same paper she investigated several results on this newly defined concept. Gracefulness and odd gracefulness are two entirely different concepts. A graph may possess one or both of these or neither. Kathiresan has discussed odd gracefulness of ladders and the graphs obtained from them by subdividing each step exactly once. Sekar has proved that the splitting graph of path $P_n$ and the splitting graph of even cycle $C_n$ are odd graceful graphs. In the present work we investigate that every α-graceful graph is also odd graceful. And using graph operation union for different graph families we show that they are also odd graceful.

## II. Main Results

**Theorem 1:** Every α- graceful graph is also odd graceful.

**Proof:** Let $G$ be an α- graceful graph and $f: V(G) \rightarrow \{0, 1, \ldots, q\}$ be an injective α- graceful labeling for $G$, where $q = |E(G)|$. Since, $G$ is α- graceful with the α- graceful labeling $f$ on $G$, $\exists\ v_1, v_2 \subseteq V(G)$ and a non-negative integer $k\ (0 \leq k < q)$ such that $v_1 = \{v \in V(G) / f(v) \leq k\}$, $v_2 = V(G) - v_1$ and for every $uv \in E(G), \min\{f(u), f(v)\} \leq k < \max\{f(u), f(v)\}$.

We define a vertex labeling function $g: V(G) \rightarrow \{0, 1, 2, \ldots, 2q - 1\}$ as follows:

$$g(w) = \begin{cases} 2f(w), & \text{when } w \in v_1 \\ 2f(w) - 1, & \text{when } w \in v_2 \end{cases}$$

i.e., $g = 2f$ on $v_1$ and $g = 2f - 1$ on $v_2$. Since, $f$ is an injective map, $g$ is also injective.

Moreover, for any $uv \in E(G)$,

$$\begin{aligned} g^*(uv) &= |g(u) - g(v)| \\ &= g(u) - g(v), \text{assuming } u \in v_2 \text{ and } v \in v_1 \\ &= 2f(u) - 1 - 2f(v) \\ &= 2\left|f(u) - f(v)\right| - 1 \\ &= 2f^*(uv) - 1, \end{aligned}$$

where $f^*: E(G) \rightarrow \{1, 2, \ldots, q\}$ is edge induced function on $E(G)$, by the α- graceful labeling $f$ and $g^*: E(G) \rightarrow \{1, 3, 5, \ldots, 2q - 1\}$ edge induced function on $E(G)$, by the labeling $g$ on $G$.

Since, $g^* = 2f^* - 1$, $f^*$ and $g^*$ both have same domain sets, as well as $f^*$ and $g^*$ both have codomain sets with same cardinality, $g^*$ is also a bijective map like $f^*$. Thus, $G$ admits an odd graceful labeling $g$ and hence, it is an odd graceful graph.

**Theorem 2:** $P_n \times P_m \cup P_r$ is odd graceful, for every $m, n, r \in \mathbb{N} - \{1\}$.

**Proof:** Let $G$ be a graph obtained by union of grid graph $P_n \times P_m$ and path $P_r$ of $r - 1$ length. It is obvious that $|V(P_n \times P_m)| = mn$, $|E(P_n \times P_m)| = 2mn - (m + n)$, $|V(P_r)| = r$ and $|E(P_r)| = r - 1$. Hence $|V(g)| = mn + r$ and $|E(G)| = 2mn + r - (m + n)$. Take $q = |E(G)|$, $V(P_r) = \{v_1, v_2, \ldots, v_r\}$, $V(P_n \times P_m) = \{u_{i,j} / 1 \leq i \leq n, 1 \leq j \leq m\}$, $E(P_r) = \{v_k v_{k+1} / k = 1, 2, \ldots, r - 1\}$ and $E(P_n \times P_m) = \{u_{i,j} u_{t,s} /$ either $i = t$ and $|j - s| = 1$ *or* $j = s$ and $|i - t| = 1$, $\forall i, t \in \{1, 2, \ldots, n\}$, $\forall j, s \in \{1, 2, \ldots, m\}\}$.

We define vertex labeling function $f: V(G) \rightarrow \{0, 1, 2, \ldots, 2q - 1\}$ as follows:

$$f(u_{i,1}) = \begin{cases} 2q - i, & \text{when } i \equiv 1 \ (mod\ 2) \\ i - 2, & \text{when } i \equiv 0 (mod\ 2), \forall\ i = 1,2,3, \dots, n; \end{cases}$$

$$f(u_{i,2}) = \begin{cases} 2n + i - 3, & \text{when } i \text{ is odd} \\ 2q - (2n + i) + 1, & \text{when } i \text{ is even}, \forall i = 1,2, \dots, n; \end{cases}$$

$$f(u_{i,j}) = f(u_{i,j-2}) + (2n-1)(-1)^{i+j-1}, \forall i = 1,2, \dots, n \text{ and } \forall j = 3,4,5, \dots, m;$$

$$f(v_k) = \begin{cases} ml - k, & \text{when } k \equiv 1 \ (mod\ 2) \\ ms + k - 1, & \text{when } k \equiv 0 (mod\ 2), \forall k = 1,2,3, \dots, r. \end{cases}$$

where $ml = \max\{f(u_{n,m}), f(u_{n,m-1}), f(u_{n-1,m})\}$ and $ms = \min\{f(u_{n,m}), f(u_{n,m-1}), f(u_{n-1,m})\}$. Above defined labeling pattern gives $f$ is an injective map and $f^*: E(G) \to \{1,3,5, \dots, 2q-1\}$ defined by $f^*(uv) = |f(u) - f(v)|, \forall uv \in E(G)$is a bijection. Therefore, $G$ admits an odd graceful labeling and so, it is an odd graceful graph.

**Illustration:** The odd graceful labeling of $P_7 \times P_3 \cup P_{10}$ is shown in figure-1.

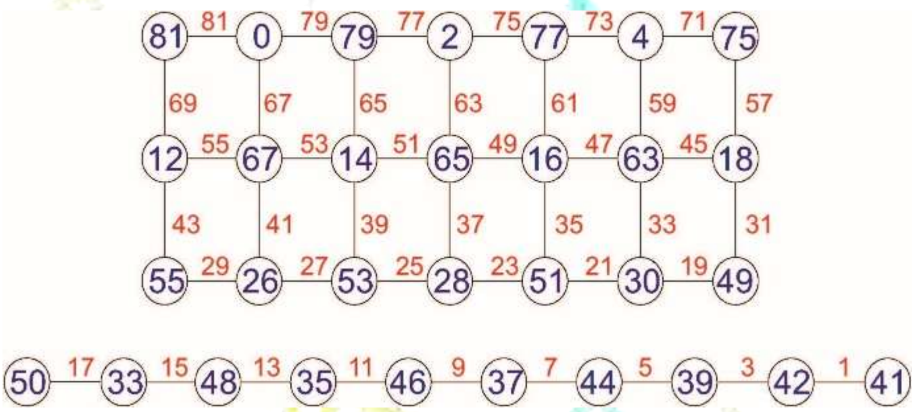


**Figure 1**

**Theorem 3:** $K_{m,n} \cup P_r$ is odd graceful, for every $m, n, r \in \mathbb{N} - \{1\}$.

**Proof:** Let $G$ be a graph obtained by union of a complete bipartite graph $K_{m,n}$ and path $P_r$. It can be observed that $|V(K_{m,n})| = m + n, |V(P_r)| = r, |E(K_{m,n})| = mn$ and $|E(P_r)| = r - 1$. Hence, $|V(G)| = m + n + r$ and $|E(G)| = mn + r - 1$. Take, $q = |E(G)|$, $V(P_r) = \{v_k / 1 \le k \le r\}$, $V(K_{m,n}) = \{w_1, w_2, \dots, w_m\} \cup \{u_1, u_2, \dots, u_n\}$ and $E(G) = \{v_k v_{k+1} / 1 \le k < r\} \cup \{w_i u_j / 1 \le i \le m, 1 \le j \le n\}$.

we define vertex labeling function $f: V(G) \to \{0,1,2, \dots, 2q-1\}$ as follows:

$$f(w_i) = 2(i-1), \forall i = 1,2, \dots, m;$$

$$f(u_j) = 2q - [2m(j-1) + 1], \forall j = 1,2, \dots, n;$$

$$f(v_k) = \begin{cases} ml - k, & \text{when } k \text{ is odd} \\ ms + k - 1, & \text{when } k \text{ is even} \end{cases} ; \forall k = 1,2,3, \dots, r.$$

where $ml = f(u_n) - 1$ and $ms = f(w_n) + 1$. Above defined labeling pattern gives $f$ is injective map and an edge induced function $f^*: E(G) \to \{1,3,5, \dots, 2q-1\}$ defined by $f^*(uv) = |f(u) - f(v)|, \forall uv \in E(G)$ is a bijection. Therefore, $g$ is an odd graceful labeling on $G$ and so, $G$ is an odd graceful graph.

**Illustration:** The odd graceful labeling of $K_{5,3} \cup P_8$ is shown in figure-2.

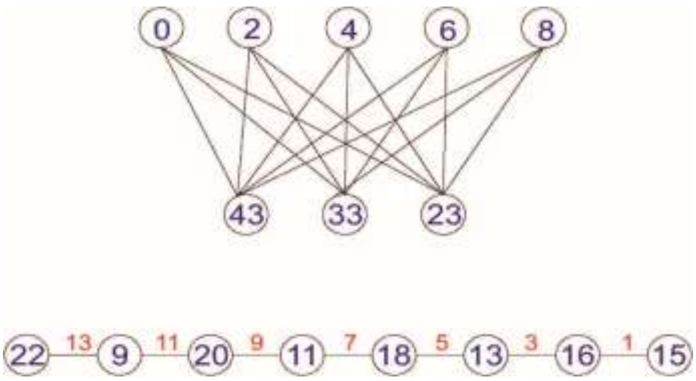


**Figure-2**

**Theorem 4:** $P_m \cup P_n$ is odd graceful, for every $m, n \in \mathbb{N} - \{1\}$.

**Proof:** Let $G$ be a graph obtained by union of two paths $P_m$ and $P_n$. Let $V(G) = \{v_1, v_2, \ldots, v_m\} \cup \{u_1, u_2, \ldots, u_m\}$ and $E(G) = \{v_i v_{i+1} / 1 \le i < m\} \cup \{u_j u_{j+1} / 1 \le j < n\}$. Take, $q = m + n - 2$ and we define vertex labeling function $f: V(G) \to \{0,1,2, \ldots, 2q - 1\}$ as follows:

$$f(v_i) = \begin{cases} 2q - i, \text{when } i \text{ is odd} \\ i - 2, \text{when } i \text{ is even} \end{cases}; \ \forall\, i = 1,2,3, \ldots, m$$

$$f(u_j) = \begin{cases} ml - j, \text{when } j \text{ is odd} \\ ms + j - 1, \text{when } j \text{ is even} \end{cases}; \ \forall\, j = 1,2,3, \ldots, n$$

where $ml = \max\{f(v_m), f(v_{m-1})\}$ *and* $ms = \min\{f(v_m), f(v_{m-1})\}$. Above defined labelling pattern gives function $f$ is an injective map. Also, its edge induced function $f^*: E(G) \to \{1,3,5, \ldots, 2q - 1\}$ defined by $f^*(uv) = |f(u) - f(v)|, \forall uv \in E(G)$ is a bijective map. Therefore, $P_m \cup P_n$ admits an odd graceful labeling and so, it is an odd graceful graph, for every $m, n \in \mathbb{N} - \{1\}$.

**Illustration:** The odd graceful labeling of $P_5 \cup P_8$ is shown in figure-3.

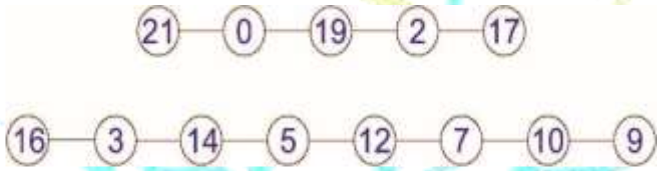


**Figure-3**

**Theorem 5:** $K_{m,n} \cup P_r \cup P_t$ is odd graceful, for every $m, n, r, t \in \mathbb{N} - \{1\}$.

**Proof:** Let $H$ be a graph obtained by union of a complete bipartite graph $K_{m,n}$ and two paths $P_r$ and $P_t$ i.e. $V(H) = V(K_{m,n}) \cup V(P_r) \cup V(P_t)$ and $E(H) = E(K_{m,n}) \cup E(P_r) \cup E(P_t)$, where $V(K_{m,n}) = \{w_{1,} w_{2,} \ldots, w_m\} \cup \{u_{1,} u_{2,} \ldots, u_n\}$ like theorem-3.

Take $q_1 = r + t - 2$, $q_2 = mn$ and $q = q_1 + q_2 = |E(H)|$. In theorem - 4, we proved that union of two paths is also an odd graceful graph. Let $h_1: V(P_r \cup P_t) \to \{0,1,2, \ldots, 2q_1 - 1\}$ be an odd graceful labeling on $P_r \cup P_t$. Since, $K_{m,n}$ is α-graceful graph, by theorem-1 it is also an odd graceful graph. Let $h_2: V(K_{m,n}) \to \{0,1,2, \ldots, 2q_2 - 1\}$ be an odd graceful labelling on $K_{m,n}$ with $h_2(w_i) = 2(i - 1), \forall i = 1,2, \ldots, m$ and $f(u_j) = 2q - [2m(j - 1) + 1], \forall j = 1,2, \ldots, n$.

We define vertex labeling function $h: V(H) \rightarrow \{0,1,2,\ldots,2q-1\}$ as follows:

$$h = \begin{cases} h_2 \text{ on } \{w_1, w_2, \ldots, w_m\} \\ h_2 + 2q_1 \text{ on } \{u_1, u_2, \ldots, u_n\} \\ h_1 + h_2(w_m) + 1 \text{ on } V(P_r \cup P_t) \end{cases}$$

Above defined labeling pattern gives $h$ is an injective map and its edge induced function $h^*: E(H) \rightarrow \{1,3,5,\ldots,2q-1\}$ defined by $h^*(uw) = |f(u) - f(w)|, \forall uw \in E(H)$ is a bijection. Therefore, $H$ admits an odd graceful labeling $h$ and so, it is an odd graceful graph.

**Illustration:** The odd graceful labeling of $K_{6,4} \cup P_{21} \cup P_{16}$ is shown in figure-4.

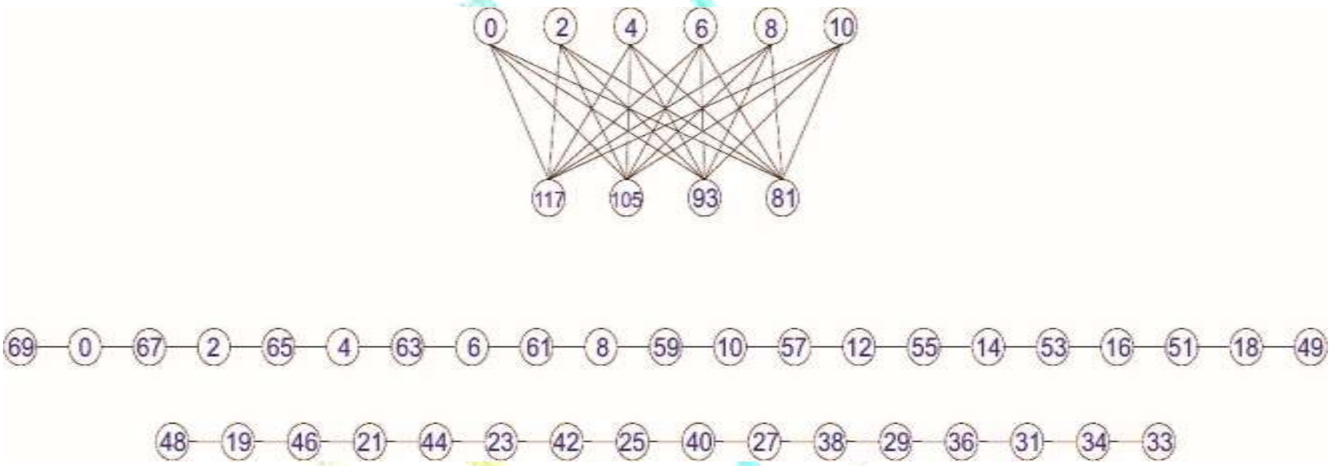


**Figure-4**

## III. Concluding Remarks

In this paper, the authors have investigated the odd graceful labeling of $\alpha$ -graceful graphs, odd graceful labeling of union of some graphs. Similar study can be extended for other graphs.

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