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Applications of Pettis Integrable Sequence of Functions Which are Not Birkhoff Integrable

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Abstract: *For X be a weakly Lindelöf determined Banach space. We follow José Rodríguez in [20] and with a bit of small changes we show (in the same way) that if X is non-separable, then there exist a complete probability space (Ω, Σ, μ) and a bounded Pettis integrable sequence of functions $f_r: \Omega \longrightarrow X$ that is not Birkhoff integrable; when the density character of X is greater than or equal to the continuum, then the sequence f_r is defined on $[0,1]$ with the Lebesgue measure. Hence, in the particular case $X = c_0(I)$ (the cardinality of I being greater than or equal to the continuum) the sequence of functions f_r can be taken as the pointwise limit of a uniformly bounded sequence of Birkhoff integrable functions, showing that the Lebesgue's dominated convergence theorem for the Birkhoff integral does not hold in general.*

Keywords: *Pettis integral; Birkhoff integral; McShane integral; dominated convergence theorem; Markushevich basis; weakly Lindelöf determined Banach space.*

I. Introduction

Many rare extensions being widely shown of the theory of Lebesgue integration to the case of functions with values in Banach spaces, those due to Bochner [5] and Pettis [13,14,18]. The Birkhoff integral [1] and the (generalized) McShane integral [8] lie strictly between the Bochner and Pettis integrals; see [2, 4, 8, 9, 10].

It is valid and well known that if the Banach space in the range is separable, then Pettis, Birkhoff and McShane integrability coincide; see [15 Corollary 5.11] and [8, Corollary 4C]. The differences arise in the non-separable case, and the situation becomes more complicated: every Birkhoff integrable function is McShane integrable, see [9, Proposition 4], and every McShane integrable function is Pettis integrable, see [8, Theorem 1Q], but none of the reverse implications hold in general, as shown in [9, Example 8] and [10, Example 3C], respectively (more examples can be found in [12], [16] and [4]). However, for certain classes of non-separable Banach spaces some equivalences remain valid: Birkhoff and McShane integrability coincide for functions with values in spaces with weak* separable dual unit ball (equivalently, spaces that are isometric to subspaces of ℓ_∞), see [9, Theorem 10], and McShane and Pettis integrability coincide for functions defined on $[0,1]$ with values in $c_0(I)$ (I any set) or superreflexive spaces; see [4].

We give an application on the study of [20] that the differences between Birkhoff and Pettis integrability go a bit further in non-separable Banach spaces. We show that for the wide class of weakly Lindelöf determined Banach spaces (WLD) there is no hope of obtaining a general result on the equivalence of the Pettis and Birkhoff integrals. Theorem 2.1 states that for a non-separable WLD Banach space X there always exist a complete probability space (Ω, Σ, μ) and a bounded Pettis integrable sequence of functions $f_r: \Omega \longrightarrow X$ that is not Birkhoff integrable. For "bigger" spaces this result can be refined in the following way: for a WLD Banach space X with density character greater than or equal to the continuum, there always exists a bounded Pettis integrable sequence of functions $f_r: [0,1] \longrightarrow X$ that is not Birkhoff integrable (see [20]).

In addition, if I is a set of cardinality greater than or equal to the continuum, a particular case of the constructions in Theorem 2.3 gives a uniformly bounded sequence of Birkhoff integrable functions $(f_r)_n: [0,1] \longrightarrow c_0(I)$ that converges pointwise to a function $f_r: [0,1] \longrightarrow c_0(I)$ that is not Birkhoff integrable (Theorem 2.5). This means that the analogue of Lebesgue's dominated convergence theorem for the Birkhoff integral does not hold in general. This negative feature is not shared by the Bochner, McShane and Pettis integrals, for which there are limit theorems that ensure the validity of Lebesgue's theorem; see [5, 8, 10] and [14].

The main references are [7] and [18]. Here c is the cardinality of the continuum and λ stands for the Lebesgue measure on the σ-algebra $\mathcal{L}$ of Lebesgue measurable subsets of $[0,1]$. Setting $|A|$ be the cardinality of an arbitrary set A. We write X^* to denote the dual of our real Banach space X. B_X is the unit ball of X. The

density character of X, denoted by $\mathrm{dens}(X)$, is the minimal cardinality of a norm dense set in X. A Markushevich basis of X is a family $\{(x_i^r, (x_i^r)^*)\}_{i\in I} \subset X \times X^*$ such that (i) $(x_i^r)^*(x_j^r) = \delta_{i,j}$ (the Kronecker symbol) for every $i, j \in I$; (ii) $\overline{\mathrm{span}}\{x_i^r\}_{i\in I} = X$ and (iii) for each $x^r \in X \setminus \{0\}$ there exists $i \in I$ such that $(x_i^r)^*(x^r) \neq 0$.

Recall that a Banach space X is said to be WLD if (B_{X^*}, weak^*) is homeomorphic to some subset S of a cube $[-1,1]^I$, endowed with the product topology, such that for each $s \in S$ the set $\{i \in I: s(i) \neq 0\}$ is countable. The class of WLD Banach spaces contains all weakly compactly generated spaces (see [7] Theorem 11.16]) and, more generally, all weakly countably determined ones (see [6, Theorem 7.2.7]). In particular, the spaces considered in [4] for which Pettis and McShane integrability coincide ($c_0(I)$ and the superreflexive ones) are WLD. The existence of Markushevich bases in WLD Banach spaces, [19, Corollary 3.1] (alternatively see Theorem 12.50 in [7]), will be an essential tool for the proofs of Theorems 2.1 and 2.3

II. Results

For (Ω, Σ, μ) be a complete probability space. The sequence of functions f_r defined on Ω with values in a Banach space X is Birkhoff integrable (with respect to μ) if, and only if, for every $\varepsilon > 0$ there is a countable partition (B_n) of Ω in Σ such that

$$\left\| \sum_{n,r} \mu(B_n) f_r(t_n^r) - \sum_{n,r} \mu(B_n) f_r(t_n^{r\prime}) \right\| < \varepsilon$$

for arbitrary choices $t_n^r, t_n^{r\prime} \in B_n$, the series involved being unconditionally convergent, is automatically fulfilled if the sequence of f_r is bounded. We have (see [20])

Theorem 2.1. Let X be a WLD Banach space. If X is non-separable, then there exist a complete probability space (Ω, Σ, μ) and a bounded Pettis integrable sequence of functions $f_r: \Omega \longrightarrow X$ that is not Birkhoff integrable.

Proof. As pointed out in the introduction, since X is WLD, there exists a Markushevich basis of X, say $\{(y_i^r, (y_i^r)^*)\}_{i\in I}$. By [17, Corollary], we can also assume that

$$\sup_{i\in I} \|y_i^r\| \cdot \|(y_i^r)^*\| < \infty$$

Set $x_i^r = \|(y_i^r)^*\| \cdot y_i^r$ and $(x_i^r)^* = \|(y_i^r)^*\|^{-1} \cdot (y_i^r)^*$ for each $i \in I$. Then $\{(x_i^r, (x_i^r)^*)\}_{i\in I}$ is a Markushevich basis of X such that $(x_i^r)^* \in B_{X^*}$ for every $i \in I$ and $\sup_{i\in I} \|x_i^r\| < \infty$.

Since $\overline{\mathrm{span}}\{x_i^r\}_{i\in I} = X$ and X is non-separable, I is uncountable. Set $\Omega := I$. Let us consider the σ-algebra Σ in Ω made up of all sets $A \subset \Omega$ for which A or $\Omega \setminus A$ is countable, and take the following complete probability measure μ on (Ω, Σ) : $\mu(A) = 0$ if A is countable, $\mu(A) = 1$ otherwise. Define $f_r: \Omega \longrightarrow X$ by $f_r(i) = x_i^r$ for every $i \in \Omega$. Obviously f_r is bounded.

On the one hand, f_r is not Birkhoff integrable. Our proof is by contradiction. Assume that f_r is Birkhoff integrable. Then there is a countable partition (B_n) of Ω in Σ such that

$$\left\| \sum_{n} \mu(B_n) f_r(t_n^r) - \sum_{n} \mu(B_n) f_r(t_n^{r\prime}) \right\| \leq \frac{1}{2} \tag{1}$$

whenever $t_n^r, t_n^{r\prime} \in B_n$. Since Ω is an atom of μ, all the B_n's but one (say B_N) are of μ-measure 0 . Inequality (1) can now be read as

$$\begin{aligned} \frac{1}{2} \geq \sup_{i,j\in B_N} \sum_r \|f_r(i) - f_r(j)\| &= \sup_{i,j\in B_N} \sum_r \|x_i^r - x_j^r\| \\ &\geq \sup_{i,j\in B_N} \sum_r (x_i^r)^*(x_i^r - x_j^r) = \sup_{i,j\in B_N} (1 - \delta_{i,j}) \end{aligned}$$

which contradicts the fact that B_N has two distinct elements (in fact, B_N is uncountable). Therefore, f_r is not Birkhoff integrable.

On the other hand, f_r is Pettis integrable. To see this fix $(x^r)^* \in X^*$. Since X is WLD, the set $\{i \in I: (x^r)^*(x_i^r) \neq 0\}$ is countable, see [7, Proposition 12.51], and therefore $(x^r)^* f_r$ vanishes μ-almost everywhere. Since $(x^r)^* \in X^*$ is arbitrary, f_r is scalarly μ-null and, in particular, Pettis integrable. The proof is complete.

It follows that a WLD Banach space X is separable if, and only if, every (bounded) Pettis integrable function defined on a complete probability space with values in X is Birkhoff integrable. Therefore, whereas the coincidence of Pettis and Bochner integrability characterizes the finite-dimensional Banach spaces, the

coincidence of Pettis and Birkhoff integrability characterizes the separable spaces inside the class of WLD Banach spaces.

The proof of Theorem 2.3 below is inspired by some ideas in the example of a Pettis integrable function that is not Birkhoff integrable given in [12]. We first need the result isolated in Lemma 2.2. Observe that, since the Borel σ-algebra of $[0,1]$ has cardinality c, the same also holds for the collection of all countable partitions of $[0,1]$ by Borel sets, which we enumerate as $\{\Gamma_{(c-\epsilon)}: \epsilon > 0\}$. Then we have (see [20])

Lemma 2.2. There exist collections $\{A_{(c-\epsilon)}\}_{\epsilon>0}$ and $\{A'_{(c-\epsilon)}\}_{\epsilon>0}$ of countable subsets of $[0,1]$ such that

(i) $A_{(c-\epsilon)} \cap A_{(c-2\epsilon)} = \emptyset$ and $A'_{(c-\epsilon)} \cap A'_{(c-2\epsilon)} = \emptyset$ for every $\epsilon > 0$ with $\epsilon \neq 0$;

(ii) $A_{(c-\epsilon)} \cap A'_{(c-2\epsilon)} = \emptyset$ for every $\epsilon > 0$;

(iii) $A_{(c-\epsilon)} \cap E \neq \emptyset$ and $A'_{(c-\epsilon)} \cap E \neq \emptyset$ for every $\epsilon > 0$ and every $E \in \Gamma_{(c-\epsilon)}$ with $\lambda(E) > 0$.

Proof. We proceed by transfinite induction. The first step is easy, since A_0 and A'_0 can be obtained by choosing exactly one point of each member of Γ_0 of positive λ-measure (taking different points for A_0 and A'_0, of course). Suppose now that $\epsilon > 0$ and that we have already constructed two collections $\{A_{(c-2\epsilon)}\}_{\epsilon>0}$ and $\{A'_{(c-2\epsilon)}\}_{\epsilon>0}$ of countable subsets of $[0,1]$ with the following properties: (1) $A_{(c-2\epsilon)} \cap A_\gamma = \emptyset$ and $A'_{(c-2\epsilon)} \cap A'_\gamma = \emptyset$ for every $\epsilon > 0$ with $(c-2\epsilon) \neq \gamma$; (2) $A_{(c-2\epsilon)} \cap A'_\gamma = \emptyset$ for every $\epsilon > 0, \gamma < (c-\epsilon)$; (3) $A_{(c-2\epsilon)} \cap E \neq \emptyset$ and $A'_{(c-2\epsilon)} \cap E \neq \emptyset$ for every $\epsilon > 0$ and every $E \in \Gamma_{(c-2\epsilon)}$ with $\lambda(E) > 0$.

Observe that $A := \bigcup_{\epsilon>0}\left(A_{(c-2\epsilon)} \cup A'_{(c-2\epsilon)}\right)$ has cardinality $|A| < c$, because each $A_{(c-2\epsilon)} \cup A'_{(c-2\epsilon)}$ is countable and $|c-\epsilon| < c$. Since each uncountable Borel subset of $[0,1]$ has cardinality c, see [3, Theorem 8.3.6], it follows that if $E \in \Gamma_{(c-\epsilon)}$ and $\lambda(E) > 0$, then $|E \setminus A| = c$ and, in particular, $E \setminus A$ has two distinct elements. Therefore, $A_{(c-\epsilon)}$ and $A'_{(c-\epsilon)}$ can be constructed by choosing exactly one element of $E \setminus A$ for every $E \in \Gamma_{(c-\epsilon)}$ of positive λ-measure (with different choices for $A_{(c-\epsilon)}$ and $A'_{(c-\epsilon)}$). The proof is finished.

Theorem 2.3. (see [20]) Let X be a WLD Banach space. If $\operatorname{dens}(X) \geq c$, then there exists a bounded Pettis integrable sequence of functions $f_r: [0,1] \longrightarrow X$ that is not Birkhoff integrable.

Proof. As in the proof of Theorem 2.1, X admits a Markushevich basis $\{(x_i^r, (x_i^r)^*)\}_{i\in I}$ such that $(x_i^r)^* \in B_{X^*}$ for every $i \in I$ and $\sup_{i\in I}\|x_i^r\| < \infty$.

Since $\operatorname{span}\{x_i^r\}_{i\in I} = X$ and $\operatorname{dens}(X) \geq c$, the cardinality of I is greater than or equal to c. Fix two injective maps $\phi, \psi: c \longrightarrow I$ such that $\phi(c) \cap \psi(c) = \emptyset$. Let $\{A_{(c-\epsilon)}\}_{\epsilon>0}$ and $\{A'_{(c-\epsilon)}\}_{\epsilon>0}$ be the collections of subsets of $[0,1]$ obtained in Lemma 2.2 Define $f_r: [0,1] \longrightarrow X$ by

$$f_r(t^r) = \begin{cases} x^r_{\phi((c-\epsilon))} & \text{if } t^r \in A_{(c-\epsilon)}, \epsilon > 0 \\ x^r_{\psi((c-\epsilon))} & \text{if } t^r \in A'_{(c-\epsilon)}, \epsilon > 0 \\ 0 & \text{if } t^r \notin \bigcup_{\epsilon>0}\left(A_{(c-\epsilon)} \cup A'_{(c-\epsilon)}\right). \end{cases} \tag{2}$$

Next we show that the bounded functions f_r satisfies the required properties.

To see that f_r is Pettis integrable fix $(x^r)^* \in X^*$. We already know that the set $\{i \in I: (x^r)^*(x_i^r) \neq 0\}$ is countable. Since ϕ and ψ are injective and the $A_{(c-\epsilon)}$ and $A'_{(c-\epsilon)}$ are countable, the set $\{t^r \in [0,1]: (x^r)^* f_r(t^r) \neq 0\}$ is countable. Since $(x^r)^* \in X^*$ is arbitrary, f_r is scalarly λ-null and, therefore, Pettis integrable.

To finish the proof we will show that f_r is not Birkhoff integrable by contradiction. Assume that f_r is Birkhoff integrable. Then there exists a countable partition $\Gamma = (B_n)$ of $[0,1]$ in $\mathcal{L}$ such that

$$\left\|\sum_{n,r} \lambda(B_n) f_r(t_n^r) - \sum_{n,r} \lambda(B_n) f_r\left(t_n^{r'}\right)\right\| \leq \frac{1}{2} \tag{3}$$

for arbitrary choices $t_n^r, t_n^{r'} \in B_n$. By the inner regularity of λ with respect to the Borel σ-algebra of $[0,1]$, we can suppose without loss of generality that Γ is made up of Borel sets, that is, $\Gamma = \Gamma_{(c-\epsilon)}$ for some $\epsilon > 0$.

Since $A_{(c-\epsilon)} \cap B_n \neq \emptyset$ and $A'_{(c-\epsilon)} \cap B_n \neq \emptyset$ whenever $\lambda(B_n) > 0$, for each $n \in \mathbb{N}$ there are some $t_n^r, t_n^{r'} \in B_n$ such that $\sum_n \lambda(B_n) f_r(t_n^r) = x^r_{\phi((c-\epsilon))}$ and $\sum_n \lambda(B_n) f_r(t_n^{r'}) = x^r_{\psi((c-\epsilon))}$. It follows that

$$\left\| \sum_{n,r} \lambda(B_n) f_r(t_n^r) - \sum_n \lambda(B_n) f_r(t_n^{r'}) \right\| = \left\| x^r_{\phi((c-\epsilon))} - x^r_{\psi((c-\epsilon))} \right\|$$
$$\geq \left(x^r_{\phi((c-\epsilon))} \right)^* \left(x^r_{\phi((c-\epsilon))} - x^r_{\psi((c-\epsilon))} \right) = 1,$$

which contradicts (3) and shows that f_r is not Birkhoff integrable. The proof is finished.

Observe that the functions f_r constructed in the proof of Theorem 2.3 is even universally Pettis integrable, i.e., f_r is Pettis integrable with respect to each Radon measure on $[0,1]$. This follows immediately from the boundedness of f_r and the fact that $\{t^r \in [0,1] : (x^r)^* f_r(t^r) \neq 0\}$ is countable for every $(x^r)^* \in X^*$.

As a consequence of Theorem 2.3 it turns out that, under the Continuum Hypothesis, a WLD Banach space X is separable if, and only if, every (bounded) Pettis integrable functions $f_r : [0,1] \longrightarrow X$ are Birkhoff integrable.

Corollary 2.4. (see [20]) Let X be a superreflexive Banach space. If $\mathrm{dens}(X) \geq c$, then there exists a bounded McShane integrable sequence of functions $f_r : [0,1] \longrightarrow X$ that is not Birkhoff integrable.
Proof. Since X superreflexive, Pettis and McShane integrability coincide for functions $f_r : [0,1] \longrightarrow X$; see [4]. The result now follows from Theorem 2.3.

From now on, given a set $I, \{(x_i^r, (x_i^r)^*)\}_{i \in I}$ will denote the standard Markushevich basis of $c_0(I)$ (i.e. $x_i^r(j) = \delta_{i,j}$ for every $i, j \in I$ and $(x_i^r)^*(y^r) = y^r(i)$ for every $y^r \in c_0(I)$ and every $i \in I$). Now we have the following (see [20]).

Theorem 2.5. Let I be a set of cardinality $|I| \geq c$. Then there exists a uniformly bounded sequence $(f_r)_n : [0,1] \longrightarrow c_0(I)$ of Birkhoff integrable functions that converges pointwise to a function $f_r : [0,1] \longrightarrow c_0(I)$ that is not Birkhoff integrable.

Proof. Let $f_r : [0,1] \longrightarrow c_0(I)$ be the function given by (2) associated to the basis $\{(x_i^r, (x_i^r)^*)\}_{i \in I}$. The proof of Theorem 2.3 reveals that f_r is not Birkhoff integrable.

Write $A_{(c-\epsilon)} = \{a^r_{(c-\epsilon),1}, a^r_{(c-\epsilon),2}, \ldots\}$ and $A'_{(c-\epsilon)} = \left\{a^{r'}_{(c-\epsilon),1}, a^{r'}_{(c-\epsilon),2}, \ldots\right\}$ for every $\epsilon > 0$, and define $D_n := \{a^r_{(c-\epsilon),n} : \epsilon > 0\} \cup \left\{a^{r'}_{(c-\epsilon),n} : \epsilon > 0\right\}$ and $E_n := \bigcup_{k=1}^n D_k$ for every $n \in \mathbb{N}$. Define $h_n := f_r \chi_{D_n}$ and $(f_r)_n := f_r \chi_{E_n} = \sum_{k=1}^n h_k$ for every $n \in \mathbb{N}$ (where χ_A denotes the characteristic function of the set A). Then $((f_r)_n)$ is a uniformly bounded sequence (since f_r is bounded) that converges pointwise to f_r. In order to prove that each $(f_r)_n$ is Birkhoff integrable it suffices to check that each h_n is Birkhoff integrable. To this end fix $n \in \mathbb{N}$ and $\varepsilon > 0$. Choose a finite partition $(C_k)_{1 \leq k \leq m}$ of $[0,1]$ in $\mathcal{L}$ such that $\lambda(C_k) \leq \varepsilon$ for every $1 \leq k \leq m$. Observe that if $t_k^r \in C_k$ for every $1 \leq k \leq m$, then

$$\left\| \sum_{k=1}^m \lambda(C_k) h_n(t_k^r) \right\|_\infty \leq \max_{1 \leq k \leq m} \lambda(C_k) \leq \varepsilon$$

$h_n|_{D_n} = f_r|_{D_n}$ is injective. Since $\varepsilon > 0$ is arbitrary, h_n is Birkhoff integrable. It follows that $(f_r)_n$ is Birkhoff integrable for every $n \in \mathbb{N}$, and the proof is finished. Observe that the functions $(f_r)_n$ obtained in the proof of Theorem 2.5 are even Riemann integrable, i.e., given $n \in \mathbb{N}$, for every $\varepsilon > 0$ there is a finite family (I_k) of non-overlapping closed intervals covering $[0,1]$ such that

$$\left\| \sum_{k,r} \lambda(I_k)(f_r)(t_k^r) - \sum_{k,r} \lambda(I_k)(f_r)(t_k^{r'}) \right\|_\infty < \varepsilon$$

whenever $t_k^r, t_k^{r'} \in I_k$. For a detailed account of the theory of Riemann integration in Banach spaces see [11].

We finish the work by Corollary (2.6) below (already known; see [9, Example 8]). Although this result can be deduced in the same way as we did in Corollary 2.4 (recall that Pettis and McShane integrability coincide for functions defined on $[0,1]$ with values in $c_0(I)$, [4]), the particular properties of our constructions allow us to provide a direct proof (see [20]).

Corollary 2.6. Let I be a set with cardinality $|I| \geq c$. Then there exists a bounded McShane integrable functions $f_r: [0,1] \longrightarrow c_0(I)$ that is not Birkhoff integrable.

Proof. We use the terminology of [10]. Let $f_r: [0,1] \longrightarrow c_0(I)$ be the function constructed in the proof of Theorem 2.5. We know that f_r is not Birkhoff integrable. In order to prove that f_r is McShane integrable fix $\varepsilon > 0$. For each $\epsilon > 0$ the set $A_{(c-\epsilon)}$ (resp. $A'_{(c-\epsilon)}$) is countable, and we can choose an open set $A_{(c-\epsilon)} \subset G_{(c-\epsilon)} \subset \mathbb{R}$ (resp. $A'_{(c-\epsilon)} \subset G'_{(c-\epsilon)} \subset \mathbb{R}$) such that $\lambda\left(G_{(c-\epsilon)}\right) \leq \varepsilon$ (resp. $\lambda\left(G'_{(c-\epsilon)}\right) \leq \varepsilon$). Fix any gauge $\delta: [0,1] \longrightarrow \mathbb{R}^+$such that $[t^r - \delta(t^r), t^r + \delta(t^r)] \subset G_{(c-\epsilon)}$ $\left(\text{resp.}\, [t^r - \delta(t^r), t^r + \delta(t^r)] \subset G'_{(c-\epsilon)}\right)$ whenever $t^r \in A_{(c-\epsilon)}$ (resp. $t^r \in A'_{(c-\epsilon)}$) and $\epsilon > 0$.

Now, if $\langle([a_i^r, b_i^r], t_i^r)\rangle_{1\leq i\leq n}$ is a McShane partition of $[0,1]$ subordinate to δ, we have

$$\left\|\sum_{i=1}^{n} (b_i^r - a_i^r) f_r(t_i^r)\right\|_\infty$$

$$= \left\|\sum_{\epsilon>0} \lambda\left(\bigcup_{t_i^r \in A_{(c-\epsilon)}} [a_i^r, b_i^r]\right) x^r_{\phi((c-\epsilon))} + \sum_{\epsilon>0} \lambda\left(\bigcup_{t_i^r \in A'_{(c-\epsilon)}} [a_i^r, b_i^r]\right) x^r_{\psi((c-\epsilon))}\right\|_\infty \leq \varepsilon$$

since $\bigcup_{t_i^r \in A_{(c-\epsilon)}} [a_i^r, b_i^r] \subset G_{(c-\epsilon)}$ and $\bigcup_{t_i^r \in A'_{(c-\epsilon)}} [a_i^r, b_i^r] \subset G'_{(c-\epsilon)}$ for every $\epsilon > 0$. Since $\varepsilon > 0$ is arbitrary, f_r is McShane integrable, with integral 0, and the proof is complete.

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